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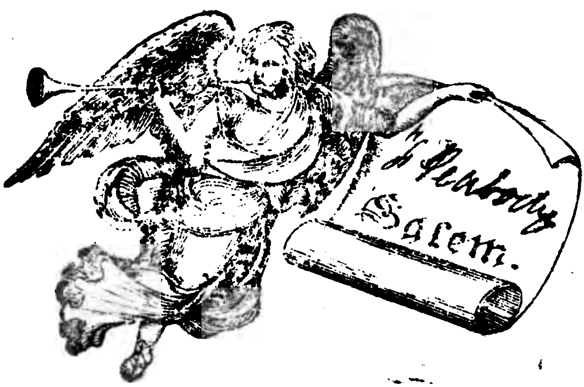
J. Peabody
Salem.

A SYSTEM OF

ARITHMETIC,

REPRINTED FROM THE

MATHEMATICAL TEXT-BOOK

COMPILED BY THE LATE

PRESIDENT WEBBER,

FOR THE USE OF

THE UNIVERSITY AT CAMBRIDGE.

CAMBRIDGE:

PUBLISHED AND SOLD BY WILLIAM HILLIARD.

1812.

Hilliard & Metcalf....printers.

DISTRICT OF MASSACHUSETTS, TO WIT;

SEAL. BE IT REMEMBERED, that on the first day of April, in the thirty sixth year of the independence of the United States of America, WILLIAM HILLIARD, of the said district, has deposited in this office the title of a book, the right whereof he claims as proprietor, in the words following, to wit; "a system of arithmetic, reprinted from the mathematical text book, compiled by the late president Webber for the use of the University at Cambridge."

In conformity to the act of the congress of the United States, entitled "an act for the encouragement of learning by securing the copies of maps, charts, and books to the authors and proprietors of such copies during the times, therein mentioned;" and also to an act, entitled "an act for the encouragement of learning by securing the copies of maps, charts, and books to the authors and proprietors of such copies during the times, therein mentioned; and extending the benefits thereof to the arts of designing, engraving, and etching historical and other prints."

W. S. SHAW, *clerk of the district of Massachusetts.*

Advertisement.

By the permission of the corporation of Harvard College, the arithmetical part of Dr. Webber's Mathematics is here published in a separate volume, with a view to the accommodation of those learners, who do not proceed to the other branches of the science; and also of instructors who are preparing scholars for the University. The examination for their admission will be conducted according to this system.

Cambridge, April 1, 1812.

CONTENTS.

ARITHMETIC.

NUMBER is the abstract ratio of one quantity to another of the same kind, taken for unity.

Theoretic Arithmetic is the science of numbers.

Practical Arithmetic is the art of numbering.

In Arithmetic there are five principal or fundamental rules for its operations, namely, Notation, Addition, Subtraction, Multiplication, and Division.

NOTATION.*

NOTATION teaches how to read any proposed number, expressed in characters, and to write any proposed number in characters.

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* As it is absolutely necessary to have a perfect knowledge of our excellent method of notation, in order to understand the reasoning made use of in the following Notes, I shall endeavour to explain it in as clear and concise a manner as possible.

1. It may then be observed, that the characters, by which all numbers are expressed, are these ten; 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; 0 is called a *cypher*, and the rest, or rather all of them, are called *figures* or *digits*. The names and signification of these characters, and the origin or generation of the numbers they stand for, are as follow; 0 nothing; 1 one, or a single

1. *To read Numbers.*

RULE.

To the simple value of each figure join the name of its place, beginning at the left and reading toward the right.

EXAMPLES.

Read the following numbers.

37	30791	111000111
101	70079	1234567890
1107	3306677	102030405060708090

---

thing, called an unit; 1 and 1 are 2 two; 2 and 1 are 3 three; 3 and 1 are 4 four; 4 and 1 are 5 five; 5 and 1 are 6 six; 6 and 1 are 7 seven; 7 and 1 are 8 eight; 8 and 1 are 9 nine; and 9 and 1 are ten, which has no single character; and thus by continual addition of one, all numbers are generated.

2. Besides the simple value of the figures, as above noted, they have each a local value according to the following law, namely, in a combination of figures, reckoning from right to left, the figure in the first place represents its primitive simple value; that in the second place, ten times its simple value; that in the third place, a hundred times its simple value, and so on; the value of the figure in each place being ten times the value of it in that immediately preceding it.

3. The names of the places are denominated according to their order. The first is called the place of units; the second, that of tens; the third, of hundreds; the fourth, of thousands; the fifth, of ten thousands; the sixth, of hundred thousands; the seventh, of millions, and so on. Thus, in the number 3456789; 9 in the first place signifies only nine; 8 in the second place signifies eight tens, or eighty; 7 in the third place is seven hundred; 6 in the fourth place is six thousand; 5 in the fifth place is fifty thousand; 4 in the sixth place is four hundred thousand; and 3 in the seventh place is three million; and the whole number is read thus, three million, four hundred and fifty six thousand, seven hundred and eighty nine.

II. *To write Numbers.*

RULE.

Write the figures in the same order as their values are expressed in, beginning at the left, and writing toward the right; remembering to supply those places of the natural order with cyphers, which are omitted in the question.

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4. A cypher, though it signifies nothing of itself, yet it occupies a place, and, when set on the right of other figures, increases their value like any other in a tenfold proportion; thus, 5 signifies only five; but 50, five tens or fifty; and 500, five hundred, &c.

5. For the more easy reading of large numbers, they are divided into periods, and half periods, each half period consisting of three figures; the name of the first period being units; that of the second, millions; of the third, billions; of the fourth trillions, &c. Also the first part of any period is the part of units; and the latter part, that of thousands.

The following Table contains a summary of the whole doctrine.

Periods.	Quadril.	Tril.	Billions.	Millions.	Units.
Half Per.	th. un.	th. un.	th. un.	th. un.	cxt cxu
Figures	123,456	789,098	765,432	101,234	567 800

A Synopsis of the Roman Notation.

1=I
2=II — As often as any character is repeated, so many
3=III — times is its value repeated.
4=IIII or IV — A less character before a greater di-
5=V — minishes its value.
6=VI — A less character after a greater in-
7=VII — creases its value.
8=VIII
9=IX

EXAMPLES.

Write in figures the following numbers.

Eighty one. Two hundred and eleven. One thousand and thirty nine. A million and a half. A hundred and four score and five thousand. Eleven thousand million, eleven hundred thousand and eleven. Thirteen billion, six hundred thousand million, four thousand and one.

EXPLANATION OF CHARACTERS.

NOTE. It may be proper to explain here certain *signs*, used in this work.

= SIGNIFIES *equality*; as, 20 shillings = 1 pound signifies, that 20 shillings are equal to one pound.

+ Signifies *plus*, or *addition;* as, 4+2=6.

— Signifies *minus*, or *subtraction;* as, 6—2=4.

× or., *Into*, signifies *multiplication;* as, 3×2 or 3·2=6.

÷ *By*, or) (signifies *division;* as, 6÷2=3, or 2)6(3.

10=X	
50=L	
100=C	
500=D or IƆ	For every Ɔ affixed this becomes 10 times as many.
1000=M or CIƆ	For every C and Ɔ, put one at each end,
2000=MM	it becomes ten times as much.
5000=IƆƆ: or $\overline{V}$	A line over any number increases it
6000=$\overline{VI}$	1000 fold.
10000=$\overline{X}$ or CCIƆƆ	
50000=IƆƆƆ	
60000=$\overline{LX}$	
100000=$\overline{C}$ or CCCIƆƆƆ	
1000000=$\overline{M}$ or CCCCIƆƆƆƆ	
2000000=$\overline{MM}$	
&c. &c.	

Division may also be denoted by placing the dividend over a line, and the divisor under it; thus $\frac{6}{2}=6\div2=3$.

.. : : .. Signifies *arithmetical proportion;* thus 2 .. 4::6 .. 8; here the meaning is, 4—2=8—6=2.

: :: : Signifies *geometrical proportion;* thus 2 : 4 :: 3 : 6, which is to be read, as 2 to 4 so is 3 to 6.

$\div$ Signifies *continual arithmetical proportion*, or *arithmetical progression;* thus, 2 : 4 : 6 : 8 $\div$ signifies, that 2, 4, 6, and 8 are in arithmetical progression.

$\dot{\div}$ Signifies *continual geometrical proportion*, or *geometrical progression;* thus, 2 : 4 : 8 : 16 $\dot{\div}$ signifies, that 2, 4, 8, 16, are in geometrical progression.

∴ Signifies *therefore.*

$\rceil^2$ Signifies *the second power*, or *square;* thus, $x\rceil^2$ signifies the square of x.

$\rceil^3$ Signifies *the third power*, or *cube.*

$\rceil^m$ Signifies *any power.*

$\surd$, or $\rceil^{\frac{1}{2}}$, Signifies *the square root;* thus $\surd x$, or $x\rceil^{\frac{1}{2}}$ signifies the square root of x.

$^3\surd$, or $\rceil^{\frac{1}{3}}$, Signifies *the cube root.*

$^n\surd$, or $\rceil^{\frac{1}{n}}$, Signifies *any root.*

$\rceil^{\frac{m}{n}}$, Signifies *any root of any power.*

The number, or letter, belonging to the above signs of powers and roots, is called *the index*, or *exponent.*

A line, called a *vinculum*, drawn over several numbers, signifies, that the numbers under it are to be considered *jointly;* thus, $20-\overline{7+8}=5$; but without the vinculum, 20—7+2=21. The same thing is also sometimes expressed by a parenthesis, inclosing two or more numbers or quantities thus, 20—(7+8)=5.

Two or more letters, joined together like those of a word, signify, that the numbers, which they represent, are to be multiplied together; thus $ab=a\times b$; and $abc=a\times b\times c$.

SIMPLE ADDITION.

Simple Addition teaches to collect several numbers of the same denomination into one number, called the *sum*.

RULE.*

1. Place the numbers under each other, so that units may stand under units, tens under tens, &c. and draw a line under them.

---

* This rule, as well as the method of proof, is founded on the known axiom, "the whole is equal to the sum of all its parts." All, that requires explaining, is the method of placing the numbers, and carrying for the tens, both which are evident from the nature of notation. For any other disposition of the numbers would entirely alter their value; and carrying one for every ten, from an inferior row or column to a superior, is evidently right, since an unit in the latter case is of the same value as ten in the former.

Beside the method here given, there is another very ingenious one of proving addition by casting out the nines.

RULE.

1. Add the figures in the first line, and find how many nines are contained in their sum.

2. Reject the nines and set the remainder in the same line, on the right.

3. Do the same in each of the other lines, and find the sum of the row of excesses. Then the nines of this sum, and of the sum of the given numbers being rejected, if the two excesses be equal, the addition is proved to be rightly performed.

EXAMPLE.

	Excesses.
3782	2
5766	6
8755	7
18303	6

2. Add the figures in the row of units, and find how many tens are contained in their sum.

3. Set the remainder under the line, and carry as many units to the next row, as there are tens, with which proceed as before; and so on till the whole is finished.

This method depends on a property of the number 9, which belongs to no other digit whatever, except 3, namely, that any number divided by 9 leaves the same remainder, as the sum of its figures or digits divided by 9; which may be thus demonstrated.

DEMON. Let there be any number, as 3467; this separated into its several parts becomes $3000+400+60+7$; but $3000=3\times1000=3\times\overline{999+1}=3\times999+3$. In like manner $400=4\times99+4$, and $60=6\times9+6$. Therefore $3467=3\times999+3+4\times99+4+6\times9+6+7=3\times999+4\times99+6\times9+3+4+6+7$. And $\frac{3467}{9}=\frac{3\times999+4\times99+6\times9}{9}+\frac{3+4+6+7}{9}$. But $3\times999+4\times99+6\times9$ is evidently divisible by 9; therefore 3467 divided by 9 will leave the same remainder, as $3+4+6+7$ divided by 9; and the same will hold for any other number whatever. Q. E. D.

The same may be demonstrated universally thus.

DEMON. Let $N=$ any number whatever, a, b, c, &c. the digits, of which it is composed, and $n=$ as many cyphers as a, the highest digit, is places from unity. Then $N=a$ with n 0s$+b$ with $\overline{n-1}$ 0s$+c$ with $\overline{n-2}$ 0s, &c. by the nature of notation; $=a\times n$ 9s$+a+b\times\overline{n-1}$ 9s$+b+c\times\overline{n-2}$ 9s$+c$, &c. $=a\times n$ 9s$+b\times\overline{n-1}$ 9s$+c\times\overline{n-2}$ 9s, &c. $+a+b+c$, &c. but $a\times n$ 9s$+b\times\overline{n-1}$ 9s$+c\times\overline{n-2}$ 9s, &c is plainly divisible by 9; therefore N divided by 9 will leave the same remainder, as $a+b+c$, &c. divided by 9. Q. E. D.

In the very same manner, this property may be shown to belong to the number three; but the preference is usually given to the number 9, on account of its being more convenient in practice.

Now from the demonstration here given, the reason of the rule itself is evident; for the excess of nines in each of two or

METHOD OF PROOF.

1. Draw a line between the first and second lines of figures to cut off the first number.

2. Add all the other numbers, and set their sum under the sum of all the numbers.

3. Add the numbers last found and the number cut off; and if their sum be the same, as that found by the first addition, the sum is right.

EXAMPLES.

(1)		(2)		(3)	
23456		22345		34578	
78901		67890		3750	
23456		8752		87	
78901		340		328	
23456		350		17	
78901		78		327	
307071	Sum.	99755	Sum.	39087	Sum.
283615		77410		4509	
307071	Proof.	99755	Proof.	39087	Proof.

---

more numbers being taken, and the excess of nines also in the sum of these excesses, it is plain, the last excess must be equal to the excess of nines, contained in the sum of all the numbers; the parts being equal to the whole.

This rule was first given by Dr. WALLIS in his Arithmetic, published A. D. 1657, and is a very simple, easy method; though it is liable to this inconvenience, that a wrong operation may sometimes appear to be right. For if we change the places of any two figures in the sum, it will still be the same. A true sum will however always appear to be true by this proof; and to make a false one appear true, there must be at least two

4. Add 8635, 2194, 7421, 5063, 2196, and 1245 together.
Answer 26754.

5. Add 246034, 298765, 47321, 58653, 64218, 5376, 9821, and 340 together. Ans. 730528.

6. Add 562163, 21964, 66321, 18536, 4340, 279, and 83 together. Ans. 663686.

7. How many days are there in the twelve calendar months? Ans. 365.

3. How many days are there from the 19th day of April, 1774, to the 27th day of November, 1775, both days exclusive? Ans. 586.

SIMPLE SUBTRACTION.

Simple Subtraction teaches to take a less number from a greater of the same denomination, and thereby shows the difference or remainder. The less number, or that which is to be subtracted, is called the *Subtrahend*; the other, the *minuend*; and the number, that is found by the operation, the *remainder* or *difference*.

RULE.*

1. Place the less number under the greater, so that units may stand under units, tens under tens, &c. and draw a line under them.

---

errors, and these opposite to each other. And if there be more than two errors, they must balance among themselves; but the chance against this particular circumstance is so great, that we may pretty safely trust to this proof.

* Demon. 1. When all the figures of the less number are less than their correspondent figures in the greater, the differences of the figures in the several like places must, taken together, make the true difference sought; because, as the sum

2. Beginning, at the right, take each figure in the subtrahend from the figure over it, and set the remainder under the line.

3. If the lower figure be greater than that over it, add ten to the upper figure; from which figure, so increased, take the lower, and write the remainder, carrying one to the next figure in the lower line, with which proceed as before; and so on till the whole is finished.

Method of Proof.

Add the remainder to the less number, and if the sum be equal to the greater, the work is right.

EXAMPLES.

(1).		(2).		(3).	
From	3287625	From	5327467	From	1234567
Take	2343756	Take	1008438	Take	345678
Rem.	943869	Remain.	4319029	Remain.	888889
Proof	3287625	Proof	5327467	Proof	1234567

---

of the parts is equal to the whole, so must the sum of the differences of all the similar parts be equal to the difference of the wholes, or given numbers.

2. When any figure of the greater number is less than its correspondent figure in the less, the ten, which is added by the rule, is the value of an unit in the next higher place, by the nature of notation; and the one, that is added to the next place of the less number, is to diminish the correspondent place of the greater accordingly; and therefore the operation in this case is only taking from one place and adding as much to another, whereby the number is never changed. And by this method the greater number is resolved into such parts, as are each greater than, or equal to the similar parts of the less; and the

4. From 2637804 take 2376982. Ans. 260822.

5. From 3762162 take 826541. Ans. 2935621.

6. From 78213606 take 27821890. Ans. 50391716.

7. The Arabian method of notation was first known in England about the year 1150; how long was it thence to the year 1776? Ans. 626 years.

8. Sir Isaac Newton was born in the year 1642, and died in 1727; how old was he at the time of his decease?

Ans. 85 years.

SIMPLE MULTIPLICATION.

Simple Multiplication is a compendious method of addition, and teaches to find the amount of any given number of one denomination, by repeating it any proposed number of times.

The number, to be multiplied, is called the *multiplicand*.

The number, to multiply, is called the *multiplier*.

The number, found from the operation, is called the *product*.

Both the multiplier and multiplicand are, in general, called *terms* or *factors*.

---

difference of the corresponding figures, taken together, will evidently make up the difference of the given numbers. Q. E. D.

The truth of the method of proof is evident; for the difference of two numbers, added to the less, is manifestly equal to the greater.

Multiplication and Division Table.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Use of the table in Multiplication.

Find the multiplier in the first column on the left, and the multiplicand in the first line; and the product is in the common angle of meeting, or against the multiplier, and under the multiplicand.

Use of the table in Division.

Find the divisor in the first column on the left, and the dividend in the same line; then the quotient will be, over the dividend, the first number of the column.

RULE.*

1. Place the multiplier under the multiplicand, so that units may stand under units, tens under tens, &c. and draw a line under them.

* DEMON. 1. When the multiplier is a single digit, it is plain, that we find the product; for by multiplying every figure, that is, every part of the multiplicand, we multiply the whole; and writing down the products, that are less than ten, and the excesses above tens respectively in the places of the figures

2. Begin at the right, and multiply the whole multiplicand severally by each figure in the multiplier, setting the first figure of every line produced directly under the figure you are multiplying by, and carrying for the tens, as in addition.

3. Add all the lines together, and their sum is the product.

---

multiplied, and carrying the number of tens in each product to the product of the next place is only gathering together the similar parts of the respective products, and is therefore the same thing, in effect, as writing the multiplicand under itself so often as the multiplier expresses, and adding the several repetitions together; for the sum of each column is the product of the figures in the place of that column; and these products, collected together, are evidently equal to the whole required product.

2. If the multiplier consists of more than one digit; having then found the product of the multiplicand by the first figure of the multiplier, as above, we suppose the multiplier divided into parts, and find, after the same manner, the product of the multiplicand by the second figure of the multiplier; but as the figure we are multiplying by stands in the place of tens; the product must be ten times its simple value; and therefore the first figure of this product must be placed in the place of tens; or, which is the same thing, directly under the figure we are multiplying by. And proceeding in this manner separately with all the figures of the multiplier, it is evident, that we shall multiply all the parts of the multiplicand by all the parts of the multiplier; or the whole of the multiplicand by the whole of the multiplier; therefore the sum of these several products will be equal to the whole required product. Q. E. D.

The reason of the method of proof depends on this proposition, namely, "that two numbers being multiplied together, either of them may be made the multiplier, or the multiplicand, and the product will be the same." A small attention to the nature of the numbers will make this truth evident; for $3\times7=$

Method of Proof.

Make the former multiplicand the multiplier, and the multiplier the multiplicand, and proceed as before; and if the product be equal to the former, the product is right.

21=7×3; and, in general, 3×4×5×6, &c.=4×3×6×5, &c. without any regard to the order of the terms; and this is true of any number of factors whatever.

The following examples are subjoined to make the reason of the rule appear as plain as possible.

(1)

37565		
5		
25	=	5×5
30	=	60×5
25	=	500×5
35	=	7000×5
15	=	30000×5
187825	=	37565×5

(2)

1375435		
4567		
9628045	=	7 times the multiplicand.
8252610	=	60 times do.
6877175	=	500 times do.
5501740	=	4000 times do.
6281611645	=	4567 times do.

Beside the preceding method of proof, there is another very convenient and easy one by the help of that peculiar property of the number 9; mentioned in addition; which is performed thus.

RULE 1. Cast the nines out of the two factors, as in addition, and write the remainder.

2. Multiply the two remainders together, and, if the excess of nines in their product be equal to the excess of nines in the total product, the answer is right.

EXAMPLES.

(1)

```
Multiply 23456787454
                by 7
         ___________
        164197512178 Product.
         ___________
```

(2)

```
Multiply  32745654473
              by  234
          _____________
           130982617892
          98236963419
         65491308946
          _____________
Product  7662483146682
```

EXAMPLE.

```
   4215  3=excess of 9s in the multiplicand.
    878  5=ditto in the multiplier.
  _____
  33720
 29505
33720
_______
3700770  6=ditto in the product=excess of 9s in 3×5.
_______
```

Demonstration of the Rule. Let M and N be the number of 9s in the factors to be multiplied, and a and b what remains; then $M+a$ and $N+b$ will be the numbers themselves, and their product is $\overline{M\times N}+\overline{M\times b}+\overline{N\times a}+\overline{a\times b}$; but the first three of these products are each a precise number of 9s, because one of their factors is so; therefore, these being cast away, there remains only $a\times b$; and if the 9s be also cast out of this, the excess is the excess of the 9s in the total product; but a and b are the excesses in the factors themselves, and $a\times b$ their product; therefore the rule is true. Q. E. D.

This method is liable to the same inconvenience with that in addition.

Multiplication may also very naturally be proved by division; for the product being divided by either of the factors, the quotient will evidently be the other; but it would have been contrary to good method to give this rule in the text, because the pupil is supposed as yet to be unacquainted with division.

3. Multiply 32745675474 by 2. Ans. 65491350948.
4. Multiply 84356745674 by 5. Ans. 421783728370.
5. Multiply 3274656461 by 12. Ans. 39295877532.
6. Multiply 273580961 by 23. Ans. 6292362103.
7. Multiply 82164973 by 3027. Ans. 248713373271.
8. Multiply 8496427 by 874359. Ans. 7428927415293.

CONTRACTIONS.

I. *When there are cyphers on the right of one or both the factors.*

RULE.

Proceed as before, neglecting the cyphers, and on the right of the product place as many cyphers as are in both the factors.

EXAMPLES.

1. Multiply 1234500 by 7500.

```
  12345
     75
  -----
  61725
 86415
 ------
 925875000 the Product.
```

2. Multiply 461200 by 72000. Ans. 33206400000.
3. Multiply 815036000 by 70300. Ans. 57297030800000.

II. *When the multiplier is the product of two or more numbers in the table.*

RULE.*

Multiply continually by those numbers or parts, instead of the whole number at once.

* The reason of this method is obvious; for any number, multiplied by the component parts of another number, must

EXAMPLES.

1. Multiply 123456789 by 25.

```
 123456789
         5
 ---------
 617283945
         5
 ---------
3086419725 the product.
```

2. Multiply 364111 by 56. Ans. 20390216.
3. Multiply 7128368 by 96. Ans. 684323328.
4. Multiply 123456789 by 1440. Ans. 177777776160.

SIMPLE DIVISION.

Simple Division teaches to find how often one number is contained in another of the same denomination, and thereby performs the work of many subtractions.

The number, to be divided, is called the *dividend*.

The number, to divide, is called the *divisor*.

The number of times, the dividend contains the divisor, is called the *quotient*.

If the dividend contain the divisor any number of times and an excess, that excess is called the *remainder*.

RULE.*

1. On the right and left of the dividend, draw a curved line, and write the divisor on the left, and the quotient, as it rises, on the right.

---

give the same product, as if it were multiplied by that number at once; thus, in example the second, 7 times the product of 8, multiplied into the given number, makes 56 times that given number, as plainly as 7 times 8 makes 56.

* According to the rule, we resolve the dividend into parts, and find by trial the number of times the divisor is contained in each of those parts; the only thing then, which remains to be

2. Find how many times the divisor may be had in so many figures of the dividend, as are just necessary, and write the number in the quotient.

3. Multiply the divisor by the quotient figure, and set the product under the part of the dividend used.

4. Subtract the last found product from that part of the dividend, under which it stands, and on the right of the remainder bring down the next figure of the dividend; which number divide as before; and proceed in this manner till the whole is finished.

---

proved, is, that the several figures of the quotient, taken as one number, according to the order, in which they are placed, is the true quotient of the whole dividend by the divisor, which may be thus demonstrated.

Demon. The complete value of the first part of the dividend is, by the nature of notation, 10, 100, or 1000, &c. times the value of which it is taken in the operation, according as there are 1, 2, or 3, &c. figures standing on the right of it; and consequently the true value of the quotient figure, belonging to that part of the dividend, is also 10, 100, or 1000, &c. times its simple value. But the true value of the quotient figure, belonging to that part of the dividend, found by the rule, is also 10, 100, or 1000, &c. times its simple value; for the number of figures on the right of it is equal to the number of remaining figures in the dividend. Therefore this first quotient figure, taken in its complete value at the place it stands in, is the true quotient of the divisor in the complete value of the first part of the dividend. For the same reason, all the rest of the figures of the quotient, taken according to their places, are each the true quotient of the divisor, in the complete value of the several parts of the dividend, belonging to each; because, as the first figure on the right of each succeeding part of the dividend has a less number of figures, by one standing on the right of it, so ought their quotients to have; and so they are actually ordered; consequently, *Method of Proof.*

Multiply the quotient by the divisor, and this product, added to the remainder, will be equal to the dividend, when the work is right.

---

all the quotient figures being taken in order as they are placed by the rule, they make one number, which is equal to the sum of the true quotients of all the several parts of the dividend; and therefore is the true quotient of the whole dividend by the divisor. Q. E. D.

To leave no obscurity in this demonstration, I shall illustrate it by an example.

EXAMPLE.

Divisor 36)85609 Dividend.

1st part of the dividend 85000
36 × 2000 = 72000 - - 2000 the 1st quotient.

1st remainder - 13000
add 600

2d part of the dividend 13600
36 × 300 = 10800 - - 300 the 2d quotient.

2d remainder - 2800
add 00

3d part of the dividend 2800
36 × 70 = 2520 - - 70 the 3d quotient.

3d remainder - 280
add 9

4th part of the dividend 289
36 × 8 = 288 - - 8 the 4th quotient.

Last remainder - 1 2378 sum of the quotients, or the answer.

EXPLANATION. It is evident, that the dividend is resolved in-

EXAMPLES.

(1)	(2)
5)13545728(2709145$\frac{3}{5}$	365)123456789(338237
10	1095
35	1395
35	1095
45	3006
45	2920
7	867
5	730
22	1378
20	1095
28	2839
25	2555
3	284

---

to these parts, 85000+600+00+9 ; for the first part of the dividend is considered only as 85, but yet it is truly 85000 ; and therefore its quotient, instead of 2, is 2000, and the remainder 13000 ; and so of the rest, as may be seen in the operation.

When there is no remainder after the operation of dividing is finished, the quotient is the absolute and perfect answer to the question ; but where there is a remainder, it may be observed, that it gives a part of another unit for the quotient, which is greater as it approaches nearer to the divisor. Thus, if the remainder be a fourth part of the divisor, the part is one fourth, or one fourth of the divisor is contained in the dividend beside the quotient already found ; if half the divisor, the part is one half, or one half of the divisor is, in addition to the quotient already found, contained in the dividend ; and so on. In order

3. Divide 3756789275474 by 2. Ans. 1878394637737.

4. Divide 12345678900 by 7. Ans. $1763668414\frac{2}{7}$.

5. Divide 9876543210 by 8. Ans. $1234567901\frac{2}{3}$.

6. Divide 1357975313 by 9. Ans. $150886145\frac{8}{9}$.

7. Divide 3217684329765 by 17. Ans. $189275548809\frac{12}{17}$.

---

therefore to complete the quotient, put the last remainder at the end of it, above a small line, and the divisor under it.

It is sometimes difficult to find how often the divisor is contained in the numbers of the several steps of the operation ; the best way will be to find how often the first figure of the divisor is contained in the first, or two first, figures of the dividend, and the answer, made less by one or two, is generally the figure wanted. Beside, if after subtracting the product of the divisor and quotient from the dividend, the remainder de equal to, or exceed the divisor, the quotient figure must be encreased accordingly.

If, when you have brought down a figure on the right of the remainder, it be still less than the divisor, a cypher must be put in the quotient, and another figure brought down, and then proceed as before.

The reason of the method of proof is plain ; for since the quotient is the number of times the dividend contains the divisor, the product of the quotient and divisor must evidently be equal to the dividend.

There are several other methods, used to prove division ; the best and most useful are the following.

Rule I. Subtract the remainder from the dividend, and divide this remainder by the quotient, and the quotient, found by this division, will be equal to the former divisor, when the work is right.

The reason of this rule is plain from what has been observed above.

8. Divide 3211473 by 27. Ans. $118943\frac{12}{27}$.

9. Divide 1406373 by 108. Ans. $13021\frac{105}{108}$.

10. Divide 293839455936 by 8405. Ans. $34960078\frac{346}{8405}$.

11. Divide 4637064283 by 57606. Ans. $80496\frac{11707}{57606}$.

Mr. Malcolm, in his Arithmetic, has fallen into an error concerning this method of proof, by making use of particular numbers, instead of a general demonstration. He says, the dividend being divided by the integral quotient, the quotient of this division will be equal to the former divisor, with the same remainder. This is true in some particular cases; but it will not hold, when the remainder is greater than the quotient, as may be easily demonstrated; but one instance will be sufficient; thus 17, divided by 6, gives the integral quotient 2, and remainder 5; but 17, divided by 2, gives the integral quotien 8, and remainder 1. This shows how cautious we ought to be in deducing general rules from particular examples.

Rule II. Add together the remainder, and all the products of the several quotient figures by the divisor, according to the order, in which they stand in the work, and the sum will be equal to the dividend, when the work is right.

The reason of this rule is extremely obvious; for the numbers, that are to be added, are the products of the divisor by each figure of the quotient separately, and each possesses, by its place, its complete value; therefore the sum of the parts, together with the remainder, must be equal to the whole.

Rule III. Subtract the remainder from the dividend, and what remains will be equal to the product of the divisor and quotient; which may be proved by casting out the nines, as was done in multiplication.

This rule has been already demonstrated in multiplication.

To avoid obscurity I shall give an example, proved according to all the different methods.

CONTRACTIONS.

I. *To divide by any number with cyphers annexed.*

RULE.*

Cut off the cyphers from the divisor, and the same number of digits from the right of the dividend; then divide,

---

EXAMPLE.

```
87)123456789(1419043          123456789
   87*             87                48
   ---        -------          --------
   364        9933301   1419043)123456741(87 Proof by Div.
   348*      11352344            11352344
   ----            48            --------
   .165       -------             9933301
   ..87*    123456789 Proof by Mult.9933301
   -----    ---------             -------
   ..786
   ..783*
     ---
   ....378
   ....348*
       ---
   .....309
   .....261*
        ---
   ......48*
   ---------
   123456789  Proof by Addition.
   ---------
```

Proof by casting out the nines.

4 is the excess of 9s in the quotient.
6 ditto - - - - in the divisor.
6 ditto - - - - in 4×6, which is also the excess of 9s in (123456741) the dividend made less by the remainder.

For illustration, we need only refer to the example; except for the proof by Addition; where it may be remarked, that the asterisms show the numbers to be added, and the dotted lines their order.

* The reason of this contraction is easily perceived; for cutting off the same figures from each is the same as dividing each of them by 10, 100, 1000, &c. and it is evident, that as often as

making use of the remaining figures, as usual, and the quotient is the answer; and what remains, written before the figures cut off, is the true remainder.

EXAMPLES.

1. Divide 310869017 by 7100.

```
71,00)3108690,17(43784 2617/7100 the quotient.
      284
      ---
       268
       213
       ---
        556
        497
        ---
         599
         568
         ---
          310
          284
          ---
          2617
```

2. Divide 7380964 by 23000. Ans. $320\frac{20964}{23000}$.

3. Divide 29628754963 by 35000. Ans. $846535\frac{20963}{35000}$.

II. *When the divisor is the product of two or more numbers in the table.*

RULE.*

Divide continually by those numbers, instead of the whole divisor at once.

---

the whole divisor is contained in the whole dividend, so often must any part of the divisor be contained in a like part of the dividend. This method is only avoiding a needless repetition of cyphers, which would happen in the common way, as may be seen by working an example at large.

* This follows from the second contraction in multiplication,

EXAMPLES.

1. Divide 31046835 by $56 = 7 \times 8$.

```
7)31046835(4435262       8)4435262(554407 the quotient.
  28                       40
  --                       --
   30                       43
   28                       40
   --                       --
    24                       35
    21                       32
    --                       --
     36                       32
     35                       32
     --                       --
      18                       62
      14                       56
      --                       --
       43                       6
       42                       --
       --
        15
        14
        --
         1
```

---

of which it is only the converse; for the third part of the half of any thing is evidently the same as the sixth part of the whole; and so of any other parts. I have omitted saying any thing in the rule about the method of finding the true remainder; for as the learner is supposed, at present, to be unacquainted with the nature of fractions, it would be improper to introduce them in this part of the work, especially as the integral quotients is sufficient to answer most of the purposes of practical division. However, as the quotient is complete without this remainder, and in some computations it is necessary it should be known, I shall here show the manner of finding it, without any assistance from fractions.

2. Divide 7014596 by 72=8×9.

```
8)7014596
 ---------
9)876824   4
 ---------
   97424   8 the quotient.
```

3. Divide 5130652 by 132. Ans. $38868\frac{76}{132}$

4. Divide 83016572 by 240. Ans. $345902\frac{62}{240}$.

RULE. Multiply the quotient by the divisor, and subtract the product from the dividend; and the result will be the true remainder.

The truth of this is extremely obvious; for if the product of the divisor and quotient, added to the remainder, be equal to the dividend, their product, taken from the dividend, must leave the remainder.

The rule, which is most commonly used, is this.

RULE. Multiply the last remainder by the preceding divisor, or last but one, and to the product add the preceding remainder; multiply this sum by the next preceding divisor, and to the product add the next proceding remainder; and so on till you have used all the divisors and remainders.

EXAMPLE.

```
9)64865 divided by 144          1 the last remainder.
 ------                   Mult. 4 the preceding divisor.
  4)7207   2                   ---
  ------                        4
  4)1801   3               Add  3 the second remainder.
  ------                       ---
     450   1                    7
                          Mult. 9 the first divisor.
                               ---
                               63
                           Add  2 the first remainder.
                               ---
Ans. 450 65/144.               65
```

To explain this rule from the example, we may observe that every unit of the first quotient may be looked upon as contain-

III. *To perform division more concisely than by the general rule.*

RULE.*

Multiply the divisor by the quotient figures as before, and subtract each figure of the product when you produce it, always remembering to carry so many to the next figure as were borrowed before.

EXAMPLES.

1. Divide 3104675846 by 833.

```
833)3104675846(3727101 713/833 the quotient.
      6056
       2257
        5915
          848
           1546
             713
```

2. Divide 29137062 by 5317. Ans. $5479\frac{5219}{5317}$.

3. Divide 62015735 by 7803. Ans. $7947\frac{5294}{7803}$.

---

ing 9 of the units in the given dividend; consequently, every unit of it, that remains, will contain the same; therefore this remainder must be multiplied by 9, in order to find the units of the given dividend, which it contains. Again, every unit in the next quotient will contain 4 units of the preceding, or 36 of the given dividend, that is, 9 times 4; therefore what remains must be multiplied by 36; or, which is the same thing, by 9 and 4 continually. Now this is the same as the rule; for instead of finding the remainders separately, they are reduced from the bottom upward, step by step, to the first, and the remaining units of the same class taken as they occur.

* The reason of this rule is the same as that of the general rule.

4. Divide 432756284563574 by 873469.

Ans. $495445498\frac{872012}{873469}$.

REDUCTION.

Reduction is the method of bringing numbers from one name or denomination to another without changing the value.

In order to perform reduction, it is necessary to be acquainted with the relative value of the different denominations of coin, weight, and measure, that are used; for which purpose see the following

TABLES OF COIN, WEIGHT, AND MEASURE.

MONEY.

4 farthings make	1 penny.	£	denotes	pounds.
12 pence	1 shilling.	*ſ* or *s*		shillings.
20 shillings	1 pound.	*d*		pence.

TROY WEIGHT.

24 grains make	1 penny-weight, marked	grs. dwt,
20 dwt.	1 ounce,	oz.
12 oz.	1 pound,	lb.

By this weight are weighed jewels, gold, silver, corn, bread, and liqours.

APOTHECARIES' WEIGHT.

20 grains make	1 scruple, marked	gr. sc. or ℈
3 sc. *or* ℈	1 dram	dr. or ʒ.
8 dr.	1 ounce	oz. or ℥.
12 oz.	1 ponnd	lb.

Apothecaries use this weight in compounding their medicines; but they buy and sell their drugs by Avoirdupois weight. Apothecaries' is the same as Troy weight, having only some different divisions.

AVOIRDUPOIS WEIGHT.

16 drams make	1 ounce, marked	dr. oz.
16 ounces	1 pound	lb.
28 lb.	1 quarter	qr.
4 quarters	1 hundred weight	cwt.
20 cwt.	1 ton	T.

By this weight are weighed all things of a coarse or drossy nature; such as butter, cheese, flesh, grocery wares, and all metals, except gold and silver.*

	lb.
* A firkin of butter is	56
A firkin of soap	64
A barrel of pot-ashes	200
A barrel of anchovies	30
A barrel of candles	120
A barrel of soap	256
A barrel of butter	224
A fother of lead is 19½ cwt.	
A stone of iron	14
A stone of butcher's meat	8

56lb. old hay } make a truss.
60lb. new hay }
36 trusses . . . . a load.

4 pecks coal make 1 bushel.
9 bushels . . . . 1 vat or strike.
36 bushels . . . . . . 1 chaldron.
21 chaldrons . . . . . . 1 score.

7lb. wool make . . . . . 1 clove.
2 cloves . . . . 1 stone.
2 stones . . . . 1 tod.
6½ tods . . . . 1 wey.
2 weys . . . . . 1 sack.
12 sacks . . . . . 1 last.

	lb.
A gallon of train oil	7½
A faggot of steel	120
A stone of glass	5
A seam of glass is 24 stone, or	120

	lb.	oz.	dr.
A peck loaf of bread weighs	17	6	1
A half peck	8	11	
A quartern	4	5	8

	lb.
A barrel of pork is	220.
A barrel of beef is	220.
A quintal of fish	112.

20 things make . . . . . 1 score.
12 . . . . 1 dozen.
12 dozen . . . . 1 gross.
144 dozed . . 1 greater gross.

Farther,—5760 grains=1 lb. Troy; 7000 grains=1 lb. Avoirdupois; therefore the weight of the pound Troy is to that of the pound Avoirdupois, as 5760 to 7000, or as 144 to 175.

DRY MEASURE.

		Marked			Marked
2 pints make	1 quart	pts. qts.	8 bushels	1 quarter	qr.
2 quarts	1 pottle	pot.	5 quarters	1 wey *or* load	wey.
2 pottles	1 gallon	gal.	4 bushels	1 coomb	co.
2 gallons	1 peck	pe.	5 pecks	1 bushel water meas.	
4 pecks	1 bushel	bu.	10 coombs	1 wey	
2 bushels	1 strike	str.	2 weys	1 last	L.

NOTE.—The diameter of a Winchester bushel is 18½ inches, and its depth 8 inches.—And one gallon by dry measure contains 268⅘ cubic inches.

By this measure, salt, lead, ore, oysters, corn, and other dry goods are measured.

ALE AND BEER MEASURE.

		Marked			Marked
2 pints make	1 quart	pts. qts.	2 firkins	1 kilderkin	kil.
4 quarts	1 gallon	gal.	2 kilderkins	1 barrel	bar.
8 gallons	1 firkin of Ale	fir.	3 kilderkins	1 hogshead	hhd.
9 gallons	1 firkin of Beer	fir.	3 barrels	1 butt	butt.

NOTE.—The ale gallon contains 282 cubic inches. In London the ale firkin contains 8 gallons, and the beer firkin 9; other measures being in the same proportion.

WINE MEASURE.

		Marked			Marked
2 pints make	1 quart	pts. qts.	2 hogshead	1 pipe *or* butt	p. *or* b.
4 quarts	1 gallon	gal.	2 pipes	1 tun	T.
42 gallons	1 tierce	tier.	18 gallons	1 runlet	rund.
63 gallons	1 hogshead	hhd.	31½ gallons	1 barrel	bar.
84 gallons	1 puncheon	pun.			

By this measure, brandy, spirits, perry, cyder, mead, vinegar, and oil are measured.

NOTE.—231 cubic inches make a gallon, and 10 gallons make an anchor.

CLOTH MEASURE.

		Marked			Marked
2¼ inches make	1 nail	nls.	3 qrs.	1 ell Flemish	Ell Fl.
4 nails	1 quarter	qrs.	5 qrs.	1 ell English	Ell Eng.
4 quarters	1 yard	yds.	6 qrs.	1 ell French	Ell Fr.

LONG MEASURE.

		Marked			Marked
3 barley corns make 1 inch		bar. c. in.	60 geographical miles, *or* 69½ stature miles	1 degree	deg. *or* °
12 inches	1 foot	ft.	360 degrees the circumference of the earth.		
3 feet	1 yard	yd.	*Note.* 4 inches make 1 hand.		
6 feet	1 fathom	fath.	5 feet	1 geometrical pace.	
5½ yards	1 pole	pol.	6 points	1 line	
40 poles	1 furlong	fur.	12 lines	1 inch.	
8 furlongs	1 mile	mls.			
3 miles	1 league	l.			

TIME.

		Marked			Marked.
60 seconds make 1 minute		s. *or* ″ m *or* ′	4 weeks	1 month	m.
60 minutes	1 hour	h. *or* °	13 months, 1 day, and 6 hours, *or* 365 days and 6 hours,	1 Julian year	Y.
24 hours	1 day	d.			
7 days	1 week	w.			

NOTE 1. The second may be suppasd to be divided into 60 thirds, and these again into 60 fourths, &c.

NOTE 2. April, June, September, and November, have each 30 days; each of the other months has 31, except February, which has 28 in common years, and 29 in leap years.

CIRCULAR MOTION.

60 seconds make	1 minute, marked	″ ′
60 minutes	1 degree	o
30 degrees	1 sign	s
12 signs, *or* 360°	1 circle.	

NEW FRENCH WEIGHTS AND MEASURES.

The weights and measures in common use are liable to great uncertainty and inconvenience. There being no fixed standard at hand, by which their accuracy can be tested, a great variety of measures, bearing the same name, has obtained in different countries. The foot, for instance, is used to stand for about thirty different established lengths in the different countries of Europe. The several denominations also of weights and measures are arbitrary, and occasion most of the trouble and perplexity, that learners meet with in mercantile arithmetic.

To remedy these evils, the French government in 1801 adopted a new system of weights and measures, the several denominations of which proceed in a decimal ratio, and all referable to a common permanent standard, established by nature, and accessible at all places on the earth. This standard is a meridian of the earth, which it was found convenient to divide into 40 million parts. One of these parts is assumed as the unity of length, and the basis of the whole system. This they called a *metre*, and is equal to about 39⅓ English inches, of which submultiples and multiples being taken, the various denominations of length are formed.

		Eng. Inch. Dec.
A millimetre *is*	the 1000th part of a metre	·03937
A centimetre	the 100th part of a metre	·39371
A decimetre	the 10th part of a metre	3·93710

I. *When the reduction is from a greater name to a less.*

RULE.*

Multiply the highest name or denomination by as many as one makes of the next less, adding to the product the parts of the second name; then multiply this sum by as many as

---

* The reason of this rule is exceedingly obvious; for pounds are brought into shillings by multiplying them by 20; shillings into pence by multiplying them by 12; and pence into farthings by multiplying them by 4; and the contrary by division; and this will be true in the reduction of numbers, containing any denominations whatever.

A METRE			39·37100
A decametre	10	metres	393·71000
A Hectometre	100	metres	3937·10000
A Kilometre	1000	metres	39371·00000
A myriometre	10000	metres	393710·00000

A grade or degree of the meridian equal to 100000 metres, or 100th part of the quadrant. 3937100.00000

	Mls.	Fur.	Yds.	Ft.	In.De.
The decametre is - - - -	0	0	10	2	9·7
The hectometre - -	0	0	109	1	1
The kilometre - - - -	0	4	213	1	10·2
The myriometre - - - - -	6	1	156	0	6
The grade or decimal degree of the meridian - - - - - -	62	1	23	2	8

MEASURES OF CAPACITY.

A cube, whose side is one tenth of a metre, that is, a cubic decimetre, constitutes the unity of measures of capacity. It is one makes of the next less name, adding to the product the parts of the third name; and so on through all the denominations to the last.

---

called a *litre*, and contains 61·028 cubic inches.

		Eng. Cub. In. Dec.
A millilitre or 1000th part of a litre	- - -	·06103
A centilitre 100th of a litre	- - - -	·61028
A decilitre 10th of a litre	- - - -	6·10280
A *litre*, a cubic decimetre	- - - -	61·02800
A decalitre 10 litres	- - - - -	610·28000
A hecatolitre 100 litres	- - - -	6102·80000
A kilalitre 1000 litres	- - - -	61028·00000
A miryolitre 10000 litres	- - - -	610280·00000

The English pint, wine measure, contains 28·875 cubic inches. The litre therefore is 2 pints and nearly one eighth of a pint.—

Hence

A decalitre is equal to 2 gal. $64\frac{44}{231}$ cubic inches.

A Hectolitre 26 gal. $4\frac{44}{231}$ cubic inches.

A kilolitre 264 gal. $\frac{44}{231}$ cubic inches.

WEIGHTS.

The unity of weight is a *gramme*. It is precisely the weight of a quantity of pure water, equal to a cubic centimetre, which is 100th of a metre, and is equal to 15·444 grains troy.

			Gr. Dec.
A milligramme is 1000th part of a gramme		- -	0·0154
A centigramme 100th of a gramme		- - -	0·1544
A decigramme 10th of a gramme		- - -	1·5444
A *gramme*, a cubic centimetre		- - - -	15·4440
A decagramme 10	grammes	- - -	154·4402
A hectogramme 100	grammes	- - -	1544·4023
A kilogramme 1000	grammes	- -	15444·0234
A myriogramme 10000	grammes	- -	154440·2344

II. *When the reduction is from a less name to a greater.*

RULE.

Divide the given number by as many as make one of the next superior denomination; and this quotient again by as

---

A gramme being equal to 15·444 grains troy.

A decagramme 6 dwt. 10·44 gr. equal to 5·65 drams avoirdupois.

		lb.	oz.	dr.	
A hectogram	equal to - -	0	3	8·5	avoird.
A kilogram	- - - - -	2	3	5	avoird.
A myriogram	- - - - -	22	1	15	avoird.

100 myriograms make a tun, wanting 32 lb. 8 oz.

LAND MEASURE.

The unity is an *are*, which is a square decametre, equal to 3·95 perches. The deciare is a tenth of the are—the centiare is 100th of an are, and equal to a square metre. The milliare is 1000th of the are. The deciare is equal to 10 ares; the hectare to 100 ares, and equal to 2 acres 1 rood 35·4 perches English. The kilare is equal to 1000 ares, the myriare to 10000 ares.

For fire-wood the *stere* is the unity of measure. It is equal to a cubic metre, containing 35·3171 cubic feet English. The decestere is the tenth of a stere.

The quadrant of the circle generally is divided like the fourth part of the meridian, into 100 degrees, each degree into 100 minutes and each minute into 100 seconds, &c. so that the same number, which expresses a portion of the meridian, indicates also its length, which is a great convenience in navigation.

The coin also is comprehended in this system, and made to conform to their unity of weight. The weight of the *Franc*, of which one tenth is alloy, is fixed at five grammes; its tenth part is called *décime*, its hundredth part *centime*.

The divisions of time soon after the adoption of the above underwent a similar alteration.

many as make one of the next following; and so on through all the denominations to the highest; and this last quotient, together with the several remainders, will be the answer required.

The method of proof is by reversing the question.

EXAMPLES.

1. In 1465l. 14s. 5d, how many farthings?

```
    20                4)1407092
  -----               ---------
  29314               12)351773
     12               ---------
  -----               2,0)2931,4 5
  351773                ---------
       4              Proof 1465l. 14s. 5d.
  -----
 1407092 the answer.
```

2. In 12l. how many farthings? Ans. 11520.

3. In 6169 pence how many pounds?

Ans. 25l. 14s. 1d.

4. In 35 guineas how many farthings?

Ans. 47840.

The year was made to consist of 12 months of 30 days each, and the excess of 5 day's in common and 6 in leap years was considered as belonging to no month. Each month was divided into three parts, called decades. The day was divided into 10 hours, each hour into 100 minutes, and each minute into 100 seconds. This new calendar was adopted in 1793; in 1805 it was abolished, and the old calendar restored. The weights and measures are still in use, and will probably soon prevail throughout the continent of Europe. They are recommended to the attention of every civilized country; and their general adoption would prove of no small importance to the scientific, as well as to the commercial world.

5. In 420 quarter guineas how many moidores?
Ans. 81⅔.

6. In 231l. 16s. how many ducats at 4s. 9d. each?
Ans. 976.

7. In 274 marks, each 17s. 9d. and 87 nobles, each 8s. 11d. how many pounds? Ans. 281l. 19s. 3d.

8. In 1776 quarter guineas how many six pences?
Ans. 24864.

9. Reduce 1776 six and thirties to half-crowns sterling.
Ans. 25574⅖.

10. In 50807 moidores how many peices of coin, each 4s. 6d.? Ans. 406456.

11. In 213210 grains how many lb.?
Ans. 37lb. 3dwt. 18gr.

12. In 59lb. 13dwts. 5gr. how many grains?
Ans. 340157.

13. In 8012131 grains how many lb.?
Ans. 1390lb. 11oz. 18dwts. 19grs.

14. In 35 tons, 17cwt. 1qr. 23lb. 7oz. 13dr. how many drams? Ans. 20571005.

15. In 37cwt. 2qr. 17lb. how many pounds Troy, a pound Avoirdupois being equal to 14oz. 11dwt. 15½grs. Troy?
Ans. 5124lb. 5oz. 10dwt. 11½grs.

16. How many barley corns will reach round the world, supposing it, according to the best calculation, to be 8340 leagues? Ans. 4755801600.

17. In 17 pieces of cloth, each 27 Flemish ells, how many yards? Ans. 344yds. 1qr.

18. How many minutes were there from the birth of Christ to the year 1776, allowing the year to consist of 365d. 5h. 48′ 58″? Ans. 934085364′ 48″.

COMPOUND ADDITION.

Compound Addition teaches to collect several numbers of different denominations into one sum.

RULE.*

1. Place the numbers so, that those of the same denomination may stand directly under each other, and draw a line under them.

2. Add the figures in the lowest denomination, and find how many ones of the next higher denomination are contained in their sum.

3. Write the remainder, and carry the ones to the next denomination; with which proceed, as before; and so on through all the denominations to the highest, whose sum must be all written; and this sum, together with the several remainders, is the whole sum required.

The method of proof is the same as in Simple Addition.

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* The reason of this rule is evident from what has been said in Simple Addition; for, in addition of money, as 1 in the pence is equal to 4 in the farthings; 1 in the shillings, to 12 in the pence; and 1 in the pounds, to 20 in the shillings; therefore, carrying as directed, is nothing more than providing a method of digesting the money, arising from each column, properly in the scale of denominations; and this reasoning will hold good in the addition of compound numbers of any description whatever.

EXAMPLES.

MONEY.

£.	s.	d.
17	13	4
13	10	2
10	17	3
8	8	7
3	3	4
	8	8
54	1	4
36	8	0
54	1	4

£.	s.	d.
84	17	5½
75	13	4¼
51	17	8¾
20	10	10¼
17	15	4½
10	10	11
261	5	8¼
176	8	2¾
261	5	8¼

£.	s.	d.
175	10	10
107	13	11¾
89	18	10
75	12	2¼
3	3	3¾
1		½
452	19	2¼
277	8	4¼
452	19	2¼

TROY WEIGHT.

lb.	oz.	dwt.	gr.
17	3	15	11
13	2	13	13
15	3	14	14
13	10		
12	1		17
		13	14

lb.	oz.	dwt.	gr.
14	10	13	20
13	10	18	21
14	10	10	10
10	1	2	3
1	4	4	4
	1	19	

lb.	oz.	dwt.	gr.
27	10	17	18
17	10	13	13
13	11	13	1
10	1		2
4	4	3	3
2			1

AVOIRDUPOIS WEIGHT.

cwt.	qr.	lb.	oz.	dr.
15	2	15	15	15
13	2	17	13	14
12	2	13	14	14
10	1	17	15	
12	1	10		10
10	1	12	1	7

T.	cwt.	qr.	lb.	oz.	dr.
2	17	3	13	8	7
2	13	3	14	8	8
1	16		10		5
2	13			1	7
1	14	1	1	2	2
4	16	1	7	7	5

T.	cwt.	qr.	lb.	oz.	dr.
3	13	2	10	7	7
2	14	1	17	6	6
4	17		14		6
2	13		12	7	7
3	13		10	4	4
5		2	12	8	3

LONG MEASURE.

Mls.	fur.	pol.	yd.	ft.	in.
37	3	14	2	1	5
28	4	17	3	2	10
17	4	4	3	1	2
10	5	6	3	1	7
29	2	2	2		3
30			4		2

Mls.	fur.	pol.	yd.	ft.	in.
28	2	13	1	1	4
39	1	17	2	2	10
28	1	14	2	2	
48	1	17	2	2	7
37	1	29			3
2		20		2	1

Mls.	fur.	pol	yd.	ft.	in.
28	3	7	2		7
30			1		7
27	6	30	2	2	
7	6	20	2	1	
5	2			2	10
	7	10		2	2

COMPOUND SUBTRACTION.

Compound Subtraction teaches to find the difference of any two numbers of different denominations.

RULE.*

1. Place the less number under the greater so, that those parts, which are of the same denomination, may stand directly under each other, and draw a line under them.

2. Beginning at the right, take the number in each denomination of the lower line from the number in the same denomination over it, and set the remainders in a line under them.

3. But if the lower number be greater than that above it, increase the upper number by as many as make one of the next higher denomination, and from this sum take the lower number and set the remainder as before.

4. Carry one for the number borrowed to the next number in the lower line, and subtract as before; and so on, till the whole is finished; and all the several remainders, taken together as one number, will be the whole difference required.

The method of proof is the same as in Simple Subtraction.

EXAMPLES.

MONEY.

	£.	s.	d.	£.	s.	d.	£.	s.	d.
From	275	13	4	454	14	2¾	274	14	2¼
Take	176	16	6	276	17	5½	85	15	7¾
Rem.	98	16	10	177	16	9¼	188	18	6½
Proof	275	13	4	454	14	2¾	374	14	2¼

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* The reason of this rule will readily appear from what has been said in Simple Subtraction; for the borrowing depends upon the very same principle, and is only different, as the numbers to be subtracted are of different denominations.

TROY WEIGHT.

	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.	lb.	oz.	dwt.	gr.
From	7	3	14	11	27	2	10	20	29	3	14	5
Take	3	7	15	20	20	3	5	21	20	7	15	7

AVOIRDUPOIS WEIGHT.

	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.	cwt.	qr.	lb.	oz.	dr.
From	5		17	5	9	22	2	13	4	8	21	1	7	6	13
Take	3	3	21	1	7	20	1	17	6	6	13		8	8	14

LONG MEASURE.

	Mls.	fur.	pol.	yd.	ft.	in.	Mls.	fur.	pol.	yd.	ft.	in.	Mls.	fur.	pol.	yd.	ft.	in.
Fr.	14	3	17	1	2	1	70	7	13	1	1	2	70	3	10			3
Ta.	10	7	80	2		10	20		14	2	2	7	17	3	11	1	1	7

TIME.

	m.	w.	d.	h.	′	m.	w.	d.	h.	′	m.	w.	d.	h.	′
From	17	2	5	17	26	37	1		13	1	71				5
Take	10			18	18	15	2		15	14	17		5	5	7
Rem.															
Proof															

COMPOUND MULTIPLICATION.

Compound Multiplication teaches to find the amount of any given number of different denominations by repeating it any proposed number of times.

RULE.*

1. Place the multiplier under the lowest denomination of the multiplicand.

2. Multiply the number of the lowest denomination by the multiplier, find how many ones of the next higher denomination are contained in the product.

3. Write the excess, and carry the ones to the product of the next higher denomination, with which proceed as before; and so on through all the denominations to the highest, whose product, together with the several excesses, taken as one number, will be the whole amount required.

The method of proof is the same as in Simple Multiplication.

EXAMPLES OF MONEY.

1. 9lb. of tobacco, at 2s. 8½d. per lb.

2s. 8½d.
9

1l. 4s. 4½. the answer.

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* The product of a number, consisting of several parts or denominations, by any simple number whatever, will evidently be expressed by taking the product of that simple number and each part by itself, as so many distinct questions; thus, 25l. 12s. 6d. multiplied by 9 will be 225l. 108s. 54d.=(by taking the shillings from the pence, and the pounds from the shillings, and placing them in the shillings and pounds respectively) 230l. 12s. 6d. which is agreeable to the rule; and this will be true, when the multiplicand is any compound number whatever.

2. 3lb. of green tea, at 9s. 6d. per pound.

Ans. 1l. 8s. 6d.

3. 5lb. of loaf sugar, at 1s. 3d. per lb.

Ans. 6l. 3s.

4. 9cwt. of cheese, at 1l. 11s. 5d. per cwt.

Ans. 14l. 2s. 9d.

5. 12 gallons of brandy, at 9s. 6d. per gallon.

Ans. 5l. 14s.

CASE 1.

If the multiplier exceed 12, multiply successively by its component parts, instead of the whole number at once, as in Simple Multiplication.

EXAMPLES.

1. 16cwt. of cheese at 1l. 18s. 8d. per cwt.

1l.	18s.	8d.
		4
7	14	8
		4
£30	18	8 the answer.

2. 28 yards of broad cloth, at 19s. 4d. per yard.

Ans. 27l. 1s. 4d.

3. 96 quarters of rye, at 1l. 3s. 4d. per quarter.

Ans. 112l.

4. 120 dozen of candles, at 5s. 9d. per doz.

Ans. 34l. 10s.

5. 132 yards of Irish cloth, at 2s. 4d. per yard.

Ans. 15l. 8s.

6. 144 reams of paper, at 13s. 4d. per ream.

Ans. 96l.

CASE 2.

If the multiplier cannot be produced by the multiplication of small numbers, find the product of such numbers nearest to it, either greater or less, then multiply by the component parts as before; and for the odd parts, add or subtract as the case requires.

EXAMPLES.

1. 17 ells of holland, at 7s. 8½d. per ell.

	7s.	8½d.
		4
1	10	10
		4
6	3	4
	7	8½
£6	11	0½ the answer.

2. 23 ells of dowlas, at 1s. 6½d. per ell.

Ans. 1l. 15s. 5½d.

3. 46 bushels of wheat, at 4s. 7¼d. per bushel.

Ans. 10l. 11s. 9½d.

4. 59 yards of tabby, at 7s. 10d. per yard.

Ans. 23l. 2s. 2d.

5. 94 pair of silk stockings, at 12s. 2d. per pair.

Ans. 57l. 3s. 8d.

6. 117 cwt. of Malaga raisins, at 1l. 2s. 3d. per cwt.

Ans. 130l. 3s. 3d.

EXAMPLES OF WEIGHTS AND MEASURES.

lb. oz. dwt. gr.	lb. oz. dr. sc. gr.	cwt. qr. lb. oz.	mls. fur. pls. yd.
21 1 7 13	2 4 2 1 0	27 1 13 12	24 3 20 2
4	7	12	6

COMPOUND DIVISION.

Compound Division teaches to find how often one number is contained in another of different denominations.

RULE.*

1. Place the numbers as in Simple Division.

2. Beginning at the left, divide each denomination by the divisor, setting the quotients under their respective dividends.

3. But if there be a remainder after dividing any of the denominations except the least, reduce it to the next lower denomination, and add to it any number, which may be in that denomination; then divide the sum as usual; and so on till the whole is finished.

The method of proof is the same as in Simple Division.

EXAMPLES OF MONEY.

1. Divide 225l. 2s. 4d. by 2.

2)225l. 2s. 4d.

112l. 11s. 2d. the quotient.

2. Divide 751l. 14s. 7½d. by 3. Ans. 250l. 11s. 6½d.

---

* To divide a number, consisting of several denominations, by any simple number whatever, is evidently the same as dividing all the parts or members, of which that number is composed, by the same simple numbers. And this will be true, when any of the parts are not an exact multiple of the divisor; for by conceiving the number, by which it exceeds that multiple, to have its proper value by being placed in the next lower denomination, the dividend will still be divided into parts, and the true quotient found as before; thus 25l. 12s. 3d. divided by 9, will be the same as 18l. 144s. 99d. divided by 9, which is equal to 2l. 16s. 11d. as by the rule; and the method of carrying from one denomination to another is exactly the same.

3. Divide 821l. 17s. 9¾d. by 4. Ans. 205l. 9s. 5¼d.

4. Divide 28l. 2s. 1½d. by 6. Ans. 4l. 13s. 8¼d.

5. Divide 135l. 10s. 7d. by 9. Ans. 15l. 1s. 2d.

6. Divide 227l. 10s. 5d. by 11. Ans. 20l. 13s. 8d.

CASE 1.

If the divisor exceed 12, divide continually by its component parts, as in Simple Division.

EXAMPLES.

1. What is cheese per cwt. if 16 cwt. cost 30l. 18s. 8d.?

```
4)30l. 18s. 8d.
 ---------------
4)7    14   8
 ---------------
£1     18   8   the answer.
```

2. If 20cwt. of tobacco come to 120l. 10s. what is that per cwt.? Ans. 6l. 6d.

3. Divide 57l. 3s. 7d. by 35.

Ans. 1l. 12s. 8d.

4. Divide 85l. 6s. by 72.

Ans. 1l. 3s. 8¼d.

5. Divide 31l. 2s. 10½d. by 99.

Ans. 6s. 3½d.

6. At 18l. 18s. per cwt. how much per lb.?

Ans. 3s. 4½d.

CASE 2.

If the divisor cannot be produced by the multiplication of small numbers, divide by Long Division.

EXAMPLES.

1. Divide 74l. 13s. 6d. by 17.

```
17)74 13 6 (4 7 10
   68
   ---
    6
   20
   ---
  133
  119
   ---
   14
   12
   ---
  174
   17
   ---
    4
```

2. Divide 23l. 15s. $7\frac{1}{4}$d. by 37. Ans. 12s. $10\frac{1}{4}$d.

3. Divide 315l. 3s. $10\frac{1}{2}$d. by 365. Ans. 17s. $3\frac{1}{4}$d.

EXAMPLES OF WEIGHTS AND MEASURES.

1. Divide 23lb. 7oz. 6dwt. 12gr. by 7.
Ans. 3lb. 4oz. 9dwt. 12gr.

2. Divide 13lb. 1oz. 2dr. 10gr. by 12.
Ans. 1lb. 1oz. 2sc. 10gr.

3. Divide 1061cwt. 2qrs. by 28.
Ans. 37cwt. 3qrs. 18lb.

4. Divide 375mls. 2fur. 7pls. 2yds. 1ft. 2in. by 39.
Ans. 9mls. 4fur. 39pls. 2ft. 8in.

5. Divide 120L. 2qrs. 1bu. 2pe. by 74.
Ans. 1L. 6qrs. 1bu. 3pe.

6. Divide 120mo. 2w. 3d. 5h. 20′ by 111.
Ans. 1mo. 2d. 10h. 12′.

DUODECIMALS.

Duodecimals are so called because they decrease by twelves, from the place of feet towards the right. Inches are sometimes called *primes*, and are marked thus ′; the next division, after inches, is called parts, or *seconds*, and is marked thus ″; the next is *thirds*, and marked thus ‴; and so on.

Duodecimals are commonly used by workmen and artificers, in finding the contents of their work.

MULTIPLICATION OF DUODECMALS; OR, CROSS MULTIPLICATION.

RULE.

1. Under the multiplicand write the same names or denominations of the multiplier, that is, feet under feet, inches under inches, &c.

2. Multiply each term in the multiplicand, beginning at the lowest, by the feet in the multiplier, and write each result under its respective term, observing to carry an unit for every 12, from each lower denomination to its next superior.

3. In the same manner multiply every term in the multiplicand by the inches in the multiplier, and set the result of each term one place father toward the right of those in the multiplicand.

4. Proceed in like manner with the seconds and all the rest of the denominations, if there be any more; and the sum of all the lines will be the product required.

Or the denominations of the several products will be as follow:

Feet by feet give feet.
Feet by primes give primes.

Feet by seconds give seconds,
&c.
Primes by primes give seconds.
Primes by seconds give thirds,
Primes by thirds give fourths,
&c.
Seconds by seconds give fourths.
Seconds by thirds give fifths.
Seconds by fourths give sixths,
&c.
Thirds by thirds give sixths.
Thirds by fourths give sevenths.
Thirds by fifths give eights,
&c.

In general thus;

When feet are concerned, the product is of the same denomination with the term multiplying the feet.

When feet are not concerned, the name of the product will be expressed by the sum of the indices of the two factors, or of the strokes over them.

EXAMPLES.

1. Multiply 10f. 4′ 5″ by 7f. 8′ 6″.

7	8	6			
72	6	11			
6	10	11	4		
	5	2	2	6	
79	11	0	6	6	Answer.

2. Multiply 4f. 7′ by 6f. 4′ 0″. Ans. 29f. 0′ 4″.
3. Multiply 14f. 9′ by 4f. 6′. Ans. 66f. 4′ 6″.
4. Multiply 4f. 7′ 8″ by 9f. 6′. Ans. 44f. 0′ 10″.

5. Multiply 7f. 8′ 6″ by 10f. 4′ 5.″

Ans. 79f. 11′ 0″ 6‴ 6iv.

6. Multiply 39f. 10′ 7″ by 18f. 8′ 4″.

Ans. 745f. 6′ 10″ 2‴ 4iv.

7. Multiply 44f. 2′ 9″ 2‴ 4iv. by 2f. 10′ 3″.

Ans. 126f. 2′ 10″ 8‴ 10iv. 11v.

8. Multiply 24f. 10′ 8″ 7‴ 5iv. by 9f. 4′ 6″.

Ans. 233f. 4′ 5″ 9‴ 6iv. 4v. 6vi.

9. Required the content of a floor 48f. 6′ long, and 24f. 3′ broad.

Ans. 1176f. 1′ 6″.

10. What is the content of a marble slab, whose length is 5f. 7′, and breadth 1f. 10′?

Ans. 10f. 2′ 10″.

11. Required the content of a ceiling, which is 43f. 3′ long, and 25f. 6′ broad.

Ans. 1102f. 10′ 6″.

12. The length of a room being 20f. its breadth 14f. 6′, and height 10f. 4′; how many yards of painting are in it, deducting a fire place of 4f. by 4f. 4′, and two windows, each 6f. by 3f. 2′? Ans. $73\frac{2}{27}$ yards.

13. Required the solid content of a wall 53f. 6′ long, 10f. 3′ high, and 2f. thick, Ans. 1310f. 9′.

VULGAR FRACTIONS.

1. *Fractions* are expressions for parts of an integer or whole. *Vulgar Fractions* are represented by two numbers, placed one above the other, with a line between them.

2. The number above the line is called the *numerator*; and that below the line, the *denominator*.

The denominator shows how many parts the integer is divided into; and the numerator shows how many of those parts are contained in the fraction.

3. A *proper fraction* is one, whose numerator is less than the denominator; as $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{9}$, &c.

4. An *improper fraction* is one, whose numerator exceeds the denominator; as $\frac{8}{5}$, $\frac{110}{21}$, &c.

5. A *single fraction* is a simple expression for any number of parts of the integer.

6. A *compound fraction* is the fraction of a fraction; as $\frac{1}{3}$ of $\frac{2}{3}$, $\frac{3}{4}$ of $\frac{5}{6}$, &c.

7. A *mixed number* is composed of a whole number and a fraction; as $8\frac{1}{3}$, $17\frac{6}{13}$, &c.

Note—Any whole number may be expressed like a fraction by writing 1 under it; as $\frac{8}{1}$.

8. The *common measure* of two or more numbers is that number, which will divide each of them without a remainder. Thus 3 is the common measure of 12 and 15; and the *greatest* number, that will do this, is called the *greatest common measure.*

9. A number, which can be measured by two or more numbers, is called their *common multiple;* and if it be the *least* number, which can be so measured, it is called their *least common multiple;* thus 30, 45, 60, and 75, are multiples of 3 and 5; but their least common multiple is 15.*

---

* A *prime number* is that, which can only be measured by an unit.

That number, which is produced by multiplying several numbers together, is called a *composite number.*

A *perfect number* is equal to the sum of all its aliquot part.

The following perfect numbers are taken from the Petersburgh acts, and are all, that are known at present.

PROBLEM 1.

To find the greatest common measure of two or more numbers.

RULE.*

1. If there be two numbers only, divide the greater by the less, and this divisor by the remainder, and so on till nothing

---

6
28
496
8128
33550336
8589869056
137438691328
2305843008139952128
2417851639228158837784576
9903520314282971830448816128

There are several other numbers, which have received different denominations, but they are principally of use in Algebra, and the higher parts of mathematics.

* The truth of the rule may be shown from the first example.—For since 54 measures 108, it also measures 108+54, or 162.

Again, since 54 measures 108, and 162, it also measures $5\times162+108$, or 918. In the same manner it will be found to measure $2\times918+162$, or 1998, and so on. Therefore 54 measures both 918 and 1998.

It is also the greatest common measure; for suppose there be a greater, then since the greater measures 918 and 1998, it also measures the remainder 162; and since it measures 162 and 918, it also measures the remainder 108; in the same manner it will be found to measure the remainder 54; that is, the greater measures the less, which is absurd. Therefere 54 is the greatest common measure.

In the very same manner, the demonstration may be applied to 3 or more numbers.

remains, always dividing the last divisor by the last remainder; then will the last divisor be the greatest common measure required.

2. When there are more than two numbers, find the greatest common measure of two of them as before; and of that common measure and one of the other numbers; then will the greatest common measure, last found, be the answer.

3. If 1 happen to be the common measure, the given numbers are prime to each other, and found to be incommeasurable.

EXAMPLES.

1. Required the greatest common measure of 918, 1998, and 522.

918)1998(2
    1836
    ——
     162)918(5
         810
         ——
         108)162(1
             108
             ——
              54)108(2
                 108
                 ——

So 54 is the greatest common measure of 1998 and 918.

Hence 54)522(9
          486
          ——
          36)54(1
             36
             ——
             18)36(2
                36
                ——

Therefore 18 is the answer required.

2. What is the greatest common measure of 612 and 540? Ans. 36.

3. What is the greatest common measure of 720, 336, and 1736? Ans. 8.

PROBLEM 2.

To find the least common multiple of two or more numbers.

RULE.*

1. If there be only two numbers, divide their product by their greatest common measure; and the quotient will be their least common multiple.

2. When there are more than two numbers, find the least common multiple of two of them as before; and of that common multiple and one of the other numbers; and so on through all the numbers to the last; then will the least common multiple, last found, be the answer.

3. If the numbers be prime to each other, their product is their last common multiple.

EXAMPLES.

1. What is the least common multiple of 3, 5, 8, and 10?

```
        3
        5
      ---
       15 the least common multiple of 3 and 5.
        8
      ---
      120 the least common multiple of 3, 5, and 8.
       10                    10)1200(120, hence 10 is the
      ---                                greatest common
  10)1200(120 the answer.                measure of 10 &
                                         1200.
```

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* The truth of this rule may in some measure be seen by an examination of the first example. It may be easily ascertained that 15 is the least number, that can be divided by 3 and 5 without a remainder; and that 120 is the least number, that can be divided by 3, 5, and 8 without a remainder; but this can also be divided by 10 without a remainder; therefore 120 appears to be the least common multiple of 3, 5, 8, and 10.

2. What is the least common multiple of 4 and 6?
Ans. 12.

3. What is the least number, that 3, 4, 8, and 12 will measure? Ans. 24.

4. What is the least number, that can be divided by the nine digits without a remainder? Ans. 2520.

REDUCTION OF VULGAR FRACTIONS.

Reduction of Vulgar Fractions is the bringing them out of one form into another, in order to prepare them for the operations of Addition, Subtraction, &c.

CASE 1.

To abbreviate or reduce fractions to their lowest terms.

RULE.*

Divide the terms of the given fraction by any number, that will divide them without a remainder, and these quo-

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* That dividing both the terms of the fraction equally by any number whatever will give another fraction, equal to the former, is evident. And if those divisions be performed as often as can be done, or the common divisor be the greatest possible, the terms of the resulting fraction must be the least possible.

NOTE 1. Any number, ending with an even number or a cypher, is divisible by 2.

2. Any number, ending with 5 or 0, is divisible by 5.

3. If the first place of any number on the right be 0, the whole is divisible by 10.

4. If the first two figures on the right of any number be divisible by 4, the whole is divisible by 4.

5. If the first three figures on the right hand of any number be divisible by 8 the whole is divisible by 8.

tients again in the same manner; and so on till it appears, that there is no number greater than 1, which will divide them, and the fraction will be in its lowest terms.

Or,

Divide both the terms of the fraction by their greatest common measure, and the quotients will be the terms of the fraction required.

EXAMPLES.

1. Reduce $\frac{144}{240}$ to its lowest terms.

$$\overset{(2)}{\frac{144}{240}}=\overset{(2)}{\frac{72}{120}}=\overset{(3)}{\frac{36}{60}}=\overset{(2)}{\frac{12}{20}}=\overset{(2)}{\frac{6}{10}}=\frac{3}{5}\text{, the answer.}$$

Or thus:

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144)240(1
    144
    ---
     96)144(1
         96
        ---
         48)96(2
            96
            --
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6. If the sum of the digits, constituting any number, be divisible by 3, or 9, the whole is divisible by 3, or 9.

7. All prime numbers, except 2 and 5, have 1, 3, 7, or 9, in the place of units; and all other numbers are composite.

8. When numbers, with the sign of Addition or Subtraction between them, are to be divided by any number, each of the numbers must be divided. Thus $\frac{4+8+10}{2}=2+4+5=11$.

9. But if the numbers have the sign of Multiplication between them, only one of them must be divided. Thus $\frac{3\times8\times10}{2\times6}$ $=\frac{3\times4\times10}{1\times6}=\frac{1\times4\times10}{1\times2}=\frac{1\times2\times10}{1\times1}=\frac{20}{1}=20$.

Therefore 48 is the greatest common measure, and $48)\frac{144}{240}=\frac{3}{5}$, the same as before.

2. Reduce $\frac{48}{272}$ to its least terms. Ans. $\frac{3}{17}$.
3. Reduce $\frac{192}{576}$ to its lowest terms. Ans. $\frac{1}{3}$.
4. Bring $\frac{825}{960}$ to its lowest terms. Ans. $\frac{55}{64}$.
5. Reduce $\frac{252}{364}$ to its least terms. Ans. $\frac{9}{13}$.
6. Reduce $\frac{5184}{6912}$ to its least terms. Ans. $\frac{3}{4}$.
7. Reduce $\frac{1344}{1536}$ to its lowest terms. Ans. $\frac{7}{8}$.
8. Abbreviate $\frac{6896800}{35700160}$ as much as possible.

Ans. $\frac{43105}{223126}$.

CASE 2.

To reduce a mixed number to its equivalent improper fraction.

RULE.*

Multiply the whole number by the denominator of the fraction, and add the numerator to the product, then that sum written above the denominator will form the fraction required.

EXAMPLES.

1. Reduce $27\frac{2}{9}$ to its equivalent improper fraction.

$$\begin{array}{r} 27 \\ 9 \\ \hline 243 \\ 2 \\ \hline 245 \\ \hline 9 \end{array}$$

* All fractions represent a division of the numerator by the denominator, and are taken altogether as proper and adequate expressions for the quotient. Thus the quotient of 2 divided by 3 is $\frac{2}{3}$; whence the rule is manifest; for if any number be multiplied and divided by the same number, it is evident the quotient must be the same as the quantity first given.

Or $\frac{\overline{27\times9+2}}{9}=\frac{245}{9}$ the answer.

2. Reduce $183\frac{5}{21}$ to its equivalent improper fraction.

Ans. $\frac{3848}{21}$.

3. Reduce $514\frac{5}{16}$ to an improper fraction. Ans. $\frac{8229}{16}$.

4. Reduce $100\frac{19}{59}$ to an improper fraction. Ans. $\frac{5919}{59}$.

5. Reduce $47\frac{3147}{8400}$ to an improper fraction.

Ans. $\frac{397947}{8400}$.

CASE 3.

To reduce an improper fraction to its equivalent whole or mixed number.

RULE.*

Divide the numerator by the denominator, and the quotient will be the whole or mixed number required.

EXAMPLES.

1. Reduce $\frac{981}{16}$ to its equivalent whole or mixed number.

```
16)981(61 5/16
   96
   --
    21
    16
    --
     5
```

Or,

$\frac{981}{16}=981\div16=61\frac{5}{16}$ the answer.

2. Reduce $\frac{56}{8}$ to its equivalent whole or mixed number.

Ans. 7.

3. Reduce $\frac{1245}{22}$ to its equivalent whole or mixed number. Ans. $56\frac{13}{22}$.

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* This rule is plainly the reverse of the former, and has its reason in the nature of common division.

4. Reduce $\frac{3848}{21}$ to its equivalent whole or mixed number. Ans. $183\frac{5}{21}$.

5. Reduce $\frac{621613}{514}$ to its equivalent whole or mixed number. Ans. $1209\frac{187}{514}$.

CASE 5.

To reduce a whole number to an equivalent fraction, having a given denominator.

RULE.*

Multiply the whole number by the given denominator, and place the product over the said denominator, and it will form the fraction required.

EXAMPLES.

1. Reduce 7 to a fraction, whose denominator shall be 9.

$7\times9=63$, and $\frac{63}{9}$ the answer.

And $\frac{63}{9}=63\div9=7$ the proof.

2. Reduce 13 to a fraction, whose denominator shall be 12. Ans. $\frac{156}{12}$.

3. Reduce 100 to a fraction, whose denominator shall be 90. Ans. $\frac{9000}{90}$.

CASE 5.

To reduce a compound fraction to an equivalent single one.

RULE.†

Multiply all the numerators together for a numerator, and all the denominators together for a denominator, and they will form the single fraction required.

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* Multiplication and Division are here equally used, and consequently the result is the same as the quantity first proposed.

† That a compound fraction may be represented by a single one is evident, since a part of a part must be equal to some part of the whole. The truth of the rule for this reduction may be shown as follows:

If part of the compound fraction be a whole or mixed number, it must be reduced to a fraction by one of the former cases.

When it can be done, divide any two terms of the fraction by the same number, and use the quotients instead thereof.

EXAMPLES.

1. Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{8}{11}$ to a single fraction.

$$\frac{2\times3\times4\times8}{3\times4\times5\times11}=\frac{192}{600}=\frac{16}{55}$$

Or,

$$\frac{2\times\not{3}\times\not{4}\times8}{\not{3}\times\not{4}\times5\times11}=\frac{16}{55} \text{ as before.}$$

2. Reduce $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{5}{8}$ to a single fraction. Ans. $\frac{1}{4}$.

3. Reduce $\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{5}{6}$ of $\frac{4}{11}$ to a single fraction. Ans. $\frac{5}{33}$.

4. Reduce $\frac{11}{12}$ of $\frac{7}{13}$ of $\frac{8}{19}$ of 10 to a single fraction. Ans. $\frac{1540}{741}$.

CASE 6.

To reduce fractions of different denominators to equivalent fractions, having a common denominator.

RULE 1.*

Multiply each numerator into all the denominators, ex-

Let the compound fraction to be reduced be $\frac{2}{3}$ of $\frac{4}{7}$. Then $\frac{1}{3}$ of $\frac{4}{7}=\frac{4}{7}\div3=\frac{4}{21}$, and consequently $\frac{2}{3}$ of $\frac{4}{7}=\frac{4}{21}\times2=\frac{8}{21}$, the same as by the rule, and the like will be found to be true in all cases.

If the compound fraction consist of more numbers than 2, the first two may be reduced to one, and that one and the third will be the same as the fraction of two numbers; and so on.

* By placing the numbers multiplied properly under one an-

cept its own, for a new numerator; and all the denominators continually for the common denominator.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{4}{7}$ to equivalent fractions, having a common denominator.

$1\times5\times7=35$ the new numerator for $\frac{1}{2}$
$3\times2\times7=42$ do. for $\frac{3}{5}$
$4\times2\times5=40$ do. for $\frac{4}{7}$
$2\times5\times7=70$ the common denominator.

Therefore the new equivalent fractions are $\frac{35}{70}$, $\frac{42}{70}$, and $\frac{40}{70}$, the answer.

2. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, and $\frac{7}{8}$ to fractions, having a common denominator. Ans. $\frac{144}{288}$, $\frac{192}{288}$, $\frac{240}{288}$, $\frac{252}{288}$.

3. Reduce $\frac{1}{3}$, $\frac{3}{4}$ of $\frac{4}{5}$, $5\frac{1}{2}$, and $\frac{2}{19}$ to a common demominator. Ans. $\frac{190}{570}$, $\frac{342}{570}$, $\frac{3135}{570}$ $\frac{60}{570}$.

4. Reduce $\frac{11}{13}$, $\frac{3}{4}$ of $1\frac{1}{4}$, $\frac{9}{11}$, and $\frac{5}{7}$ to a common denominator. Ans. $\frac{13552}{16016}$, $\frac{15015}{16016}$, $\frac{13104}{16016}$, $\frac{11440}{16016}$.

RULE 2.

To reduce any given fractions to others, which shall have the least common denominator.

1. Find the least common multiple of all the denomina-

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other, it will be seen, that the numerator and denominator of every fraction are multiplied by the very same number and consequently their values are not altered. Thus in the first example:

$$\frac{1}{2}\left|\frac{\times5\times7}{\times5\times7}\right. \quad \frac{3}{5}\left|\frac{\times2\times7}{\times2\times7}\right. \quad \frac{4}{7}\left|\frac{\times2\times5}{\times2\times5}\right.$$

In the 2d rule, the common denominator is a multiple of all the denominators, and consequently will divide by any of them; it is therefore manifest that proper parts may be taken for all the numerators required.

tors of the given fractions, and it will be the common denominator required.

2. Divide the common denominator by the denominator of each fraction, and multiply the quotient by the numerator, and the product will be the numerator of the fraction required.

EXAMPLES.

1. Reduce $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ to fractions, having the least common denominator.

$$\begin{array}{l} 2 \\ 3 \\ \hline 6 \end{array}$$ the least common denominator.

$6 \div 2 \times 1 = 3$ the first numerator; $6 \div 3 \times 2 = 4$ the second numerator: $6 \div 6 \times 5 = 5$ the third numerator.

Whence the required fractions are $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$.

2. Reduce $\frac{7}{12}$ and $\frac{11}{18}$ to fractions, having the least common denominator. Ans. $\frac{21}{36}$, $\frac{22}{36}$.

3. Reduce $\frac{1}{2}$, $\frac{2}{3}$ $\frac{3}{4}$, and $\frac{5}{6}$ to the least common denominator. Ans. $\frac{6}{12}$, $\frac{8}{12}$, $\frac{9}{12}$, $\frac{10}{12}$.

4. Reduce $\frac{2}{5}$, $\frac{4}{6}$, $\frac{5}{9}$, and $\frac{7}{10}$ to the least common denominator. Ans. $\frac{36}{90}$, $\frac{60}{90}$, $\frac{50}{90}$, $\frac{63}{90}$.

5. Reduce $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{16}$, and $\frac{17}{24}$ to equivalent fractions, having the least common denominator.

Ans. $\frac{16}{48}$, $\frac{36}{48}$, $\frac{40}{48}$, $\frac{42}{48}$, $\frac{33}{48}$, $\frac{34}{48}$.

CASE 7.

To find the value of a fraction in the known parts of the integer.

RULE.*

Multiply the numerator by the parts in the next inferior

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* The numerator of a fraction may be considered as a remainder, and the denominator as a divisor; therefore this rule has its reason in the nature of Compound Division.

denomination, and divide the product by the denominator; and if any thing remain, multiply it by the next inferior denomination, and divide by the denominator as before; and so on as far as necessary; and the quotients placed after one another, in their order, will be the answer required.

EXAMPLES.

1. What is the value of $\frac{5}{7}$ of a shilling?

```
   5
  12
 ----
7)60(8d. 2 4/7q. Ans.
  56
 ----
   4
   4
 ----
  16
  14
 ----
   2
```

2. What is the value of $\frac{3}{8}$ of a pound sterling?

Ans. 7s. 6d.

3. What is the value of $\frac{3}{5}$ of a pound Troy?

Ans. 7oz. 4dwt.

4. What is the value of $\frac{4}{7}$ of a pound Avoirdupois?

Ans. 9oz. $2\frac{2}{7}$dr.

5. What is the value of $\frac{7}{9}$ of a cwt.?

Ans. 3qrs. 3lb. 1oz. $12\frac{4}{9}$dr.

6. What is the value of $\frac{3}{17}$ of a mile?

Ans. 1fur. 16pls. 2yds. 1ft. $9\frac{3}{17}$in.

7. What is the value of $\frac{5}{9}$ of an ell English?

Ans. 2qrs. $3\frac{1}{9}$nls.

8. What is the value of $\frac{7}{8}$ of a tun of wine?

Ans. 3hhd. 31gal. 2qts.

9. What is the value of $\frac{7}{13}$ of a day?

Ans. 12h. 55′ $23\frac{1}{13}$″.

CASE 8.

To reduce a fraction of one denomination to that of another, retaining the same value.

RULE.*

Make a compound fraction of it, and reduce it to a single one.

EXAMPLES.

1. Reduce $\frac{5}{6}$ of a penny to the fraction of a pound.

$\frac{5}{6}$ of $\frac{1}{12}$ of $\frac{1}{20}=\frac{5}{1440}=\frac{1}{288}$ the answer.

And $\frac{1}{288}$ of $\frac{20}{1}$ of $\frac{12}{1}=\frac{240}{288}=\frac{5}{6}$d. the proof.

2. Reduce $\frac{2}{3}$ of a farthing to the fraction of a pound.

Ans. $\frac{1}{1440}$.

3. Reduce $\frac{1}{18}$l. to the fraction of a penny. Ans. $\frac{40}{3}$.

4. Reduce $\frac{4}{5}$ of a dwt. to the fraction of a pound Troy.

Ans. $\frac{1}{300}$.

5. Reduce $\frac{6}{7}$ of a pound Avoirdupois to the fraction of a cwt. Ans. $\frac{3}{392}$.

6. Reduce $\frac{9}{6552}$ of a hhd. of wine to the fraction of a pint. Ans. $\frac{9}{13}$.

7. Reduce $\frac{3}{13}$ of a month to the fraction of a day.

Ans. $\frac{84}{13}$.

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* The reason of this practice is explained in the rule for reducing compound fractions to single ones.

The rule might have been distributed into two or three different cases, but the directions here given may very easily be applied to any question, that can be proposed in those cases, and will be more easily understood by an example or two, than by a multiplicity of words.

8. *Reduce 7s. 3d. to the fraction of a pound. Ans. $\frac{29}{80}$.

9. Express 6fur. 16pls. to the fraction of a mile.

Ans. $\frac{4}{5}$.

ADDITION OF VULGAR FRACTIONS.

RULE.†

Reduce compound fractions to single ones; mixed numbers to improper fractions; fractions of different integers to those of the same; and all of them to a common denominator; then the sum of the numerators, written over the common denominator, will be the sum of the fractions required.

EXAMPLES.

1. Add $3\frac{5}{8}$, $\frac{7}{8}$, $\frac{4}{5}$ of $\frac{7}{8}$, and 7 together.

First $3\frac{5}{8}=\frac{29}{8}$, $\frac{4}{5}$ of $\frac{7}{8}=\frac{7}{10}$, $7=\frac{7}{1}$.

Then the fractions are $\frac{29}{8}$, $\frac{7}{8}$, $\frac{7}{10}$, and $\frac{7}{1}$; ∴

$$\begin{array}{r}
29\times8\times10\times1=2320\\
7\times8\times10\times1=560\\
7\times8\times8\times1=448\\
7\times8\times8\times10=4480\\
\hline
7808
\end{array}$$

$$\frac{7808}{8\times8\times10\times1=640.}=12\frac{128}{640}=12\frac{1}{5} \text{ the answer.}$$

* Thus 7s. 3d. = 87d. and 1l. = 240d. ∴ $\frac{87}{240}=\frac{29}{80}$ the answer.

† Fractions, before they are reduced to a common denominator, are entirely dissimilar, and therefore cannot be incorporated with one another; but when they are reduced to a common denominator, and made parts of the same thing, their sum or difference may then be as properly expressed by the sum or difference of the numerators, as the sum or difference of any

2. Add $\frac{5}{8}$, $7\frac{1}{2}$, and $\frac{1}{3}$ of $\frac{3}{4}$ together. Ans. $8\frac{3}{8}$.

3. What is the sum of $\frac{3}{5}$, $\frac{4}{5}$ of $\frac{1}{3}$, and $9\frac{3}{20}$? Ans. $10\frac{1}{60}$.

4. What is the sum of $\frac{9}{10}$ of $6\frac{7}{8}$, $\frac{4}{7}$ of $\frac{1}{2}$, and $7\frac{1}{2}$?

Ans. $13\frac{109}{112}$.

5. Add $\frac{1}{7}$l. $\frac{2}{9}$s. and $\frac{5}{12}$ of a penny together.

Ans. $\frac{3139}{1008}$, or 3s. 1d. $1\frac{10}{21}$.

6. What is the sum of $\frac{2}{7}$ of 15l. $3\frac{3}{7}$l. $\frac{1}{3}$ of $\frac{5}{7}$ of $\frac{3}{5}$ of a pound, and $\frac{2}{3}$ of $\frac{3}{7}$ of a shilling?

Ans. 7l. 17s. $5\frac{1}{7}$d.

7. Add $\frac{2}{3}$ of a yard, $\frac{3}{4}$ of a foot, and $\frac{3}{8}$ of a mile together.

Ans. 660yds. 2ft. 9in.

8. Add $\frac{1}{3}$ of a week, $\frac{1}{4}$ of a day, and $\frac{1}{2}$ of an hour together. Ans. 2d. $14\frac{1}{2}$h.

SUBTRACTION OF VULGAR FRACTIONS.

RULE.

Prepare the fractions as in Addition, and the difference of the numerators, written above the common denominator, will give the difference of the fractions required.

EXAMPLES.

1. From $\frac{2}{3}$ take $\frac{2}{9}$ of $\frac{3}{7}$.

$\frac{2}{9}$ of $\frac{3}{7} = \frac{2}{21}$, and $\frac{2}{3}$;

$\therefore \frac{14}{21} - \frac{2}{21} = \frac{12}{21} = \frac{4}{7}$ the answer required.

2. From $\frac{97}{100}$ take $\frac{3}{7}$. Ans $\frac{379}{700}$.

3. From $96\frac{1}{3}$ take $14\frac{3}{7}$. Ans. $81\frac{19}{21}$.

4. From $14\frac{1}{4}$ take $\frac{2}{3}$ of 19. Ans. $1\frac{7}{12}$.

5. From $\frac{1}{2}$l. take $\frac{3}{4}$s. Ans. 9s. 3d.

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two quantities whatever by the sum or difference of their individuals; whence the reason of the rules, both for Addition and Subtraction, is manifest.

6. From $\frac{3}{5}$oz. take $\frac{7}{8}$dwt. Ans. 11dwt. 3gr.

7. From 7 weeks take $9\frac{7}{10}$ days. Ans. 5w. 4d. 7h. 12′.

MULTIPLICATION OF VULGAR FRACTIONS.

RULE.*

Reduce compound fractions to single ones, and mixed numbers to improper fractions; then the product of the numerators is the numerator; and the product of the denominators, the denominator of the product required.

EXAMPLES.

1. Required the continued product of $2\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{3}$ of $\frac{5}{6}$, and 2.

$$2\frac{1}{2}\ \ \frac{5}{2},\ \frac{1}{3} \text{ of } \frac{5}{6} = \frac{1\times5}{3\times6} = \frac{5}{18}, \text{ and } 2 = \frac{2}{1};$$

Then $\frac{5}{2}\times\frac{1}{8}\ \ \frac{5}{18}\times\frac{2}{1} = \frac{5\times1\times5\times2}{2\times8\times18\times1} = \frac{25}{144}$ the answer.

2. Multiply $\frac{4}{15}$ by $\frac{5}{24}$. Ans. $\frac{1}{18}$.

3. Multiply $4\frac{1}{2}$ by $\frac{1}{8}$. Ans. $\frac{9}{16}$.

4. Multiply $\frac{1}{2}$ of 7 by $\frac{3}{6}$. Ans. $1\frac{3}{4}$.

5. Multiply $\frac{2}{9}$ of $\frac{3}{5}$ by $\frac{5}{8}$ of $3\frac{2}{7}$. Ans. $\frac{23}{84}$.

6. Multiply $4\frac{1}{2}$, $\frac{3}{4}$ of $\frac{1}{7}$, and $18\frac{4}{5}$, continually together. Ans. $9\frac{9}{140}$.

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* Multiplication by a fraction implies the taking some part or parts of the multiplicand, and therefore, may be truly expressed by a compound fraction. Thus $\frac{3}{4}$ multiplied by $\frac{5}{8}$ is the same as $\frac{3}{4}$ of $\frac{5}{8}$; and as the directions of the rule agree with the method already given to reduce these fractions to single ones, it is shown to be right.

DIVISION OF VULGAR FRACTIONS.

RULE.*

Prepare the fractions as in Multiplication; then invert the divisor, and proceed exactly as in Multiplication.

EXAMPLES.

1. Divide $\frac{1}{5}$ of 19 by $\frac{2}{3}$ of $\frac{3}{4}$.

$\frac{1}{5}$ of $19=\frac{1\times19}{5\times1}=\frac{19}{5}$, and $\frac{2}{3}$ of $\frac{3}{4}=\frac{\not{2}\times\not{3}}{\not{3}\times\not{4}}=\frac{1}{2}$;

$\therefore\ \frac{19}{5}\times\frac{2}{1}=\frac{19\times2}{5\times1}=\frac{38}{5}=7\frac{3}{5}$ the quotient required.

2. Divide $\frac{4}{7}$ by $\frac{2}{3}$. Ans. $\frac{6}{7}$.
3. Divide $9\frac{1}{6}$ by $\frac{1}{2}$ of 7. Ans. $2\frac{13}{21}$.
4. Divide $3\frac{1}{6}$ by $9\frac{1}{2}$. Ans. $\frac{1}{3}$.
5. Divide $\frac{7}{8}$ by 4. Ans. $\frac{7}{32}$.
6. Divide $\frac{1}{2}$ of $\frac{2}{3}$ by $\frac{2}{3}$ of $\frac{3}{4}$. Ans. $\frac{2}{3}$.

DECIMAL FRACTIONS.

A *Decimal* is a fraction, whose denominator is an unit, or 1, with as many cyphers annexed, as the numerator has

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* The reason of the rule may be shown thus. Suppose it were required to divide $\frac{3}{4}$ by $\frac{2}{5}$. Now $\frac{3}{4}\div2$ is manifestly $\frac{1}{2}$ of $\frac{3}{4}$ or $\frac{3}{4\times2}$; but $\frac{2}{5}=\frac{1}{5}$ of 2, $\therefore\frac{1}{5}$ of 2, or $\frac{2}{5}$ must be contained 5 times as often in $\frac{3}{4}$ as 2 is; that is $\frac{3\times5}{4\times3}=$ the answer; which is according to the rule; and will be so in all cases.

places; and is commonly expressed by writing the numerator only, with a point before it, called the *separatrix*.

Thus, 0·5 is equal to	$\frac{5}{10}$	or	$\frac{1}{2}$.
0·25	$\frac{25}{100}$	or	$\frac{1}{4}$.
0·75	$\frac{75}{100}$	or	$\frac{3}{4}$.
1·3	$\frac{13}{10}$	or	$1\frac{3}{10}$.
24·6	$24\frac{6}{10}$		
·02	$\frac{2}{100}$	or	$\frac{1}{50}$.
·0015	$\frac{15}{10000}$	or	$\frac{3}{2000}$.

A *finite* decimal is that, which ends at a certain number of places. But an *infinite* decimal is that, which is understood to be indefinitely continued.

A *repeating* decimal has one figure, or several figures, continually repeated, as far as it is found. As $\cdot 3\dot{3}$, &c. which is a *single repetend.* And $20\cdot\dot{2}4\dot{2}4$, &c. or $20\cdot 2\dot{4}6\dot{2}4\dot{6}$, &c. which are *compound repetends.* Repeating decimals are also called *circulates*, or *circulating decimals.* A point is set over a single repetend, and a point over the first and last figures of a compound repetend.

The first place, next after the decimal mark, is 10th parts, the second is 100th parts, the third is 1000th parts, and so on, decreasing toward the right by 10ths, or increasing toward the left by 10ths, the same as whole or integral numbers do. As in the following

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NOTE.—A fraction is multiplied by an integer, by dividing the denominator by it, or multiplying the numerator; and divided by an integer, by dividing the numerator, or multiplying the denominator.

SCALE OF NOTATION.

&c.	Millions.	Hundreds of thousands.	Tens of thousands.	Thousands.	Hundreds.	Tens.	Units.	Tenth parts.	Hundredth parts.	Thousandth parts.	Ten thousandth parts.	Hundred thousandth parts.	Millionth parts.	&c.
	8	8	8	8	8	8	8.	8	8	8	8	8	8	

Cyphers on the right of decimals do not alter their value.

For ·5 or $\frac{5}{10}$ is $\frac{1}{2}$.
And ·50 or $\frac{50}{100}$ is $\frac{1}{2}$.
And ·500 or $\frac{500}{1000}$ is $\frac{1}{2}$.

But cyphers before decimal figures, and after the separating point, diminish the value in a tenfold proportion for every cypher.

So ·5 is $\frac{5}{10}$ or $\frac{1}{2}$
But ·05 is $\frac{5}{100}$ or $\frac{1}{20}$
And ·005 is $\frac{5}{1000}$ or $\frac{1}{200}$
And so on.

So that, in any mixed or fractional number, if the separating point be moved one, two, three, &c. places to the right, every figure will be 10, 100, 1000, &c. times greater than before.

But if the point be moved toward the left, then every figure will be diminished in the same manner, or the whole quantity will be divided by 10, 100, 1000, &c.

ADDITION OF DECIMALS.

RULE.

1. Set the numbers under each other according to the value of their places, as in whole numbers, or so that the decimal points may stand each directly under the preceding.

2. Then add as in whole numbers, placing the decimal point in the sum directly under the other points.

EXAMPLES.

(1)

```
7530
  16·201
   3·0142
 957·13
   6·72819
    ·03014
----------
8513·10353
```

2. What is the sum of 276, 39·213, 72014·9, 417, 5032, and 2214·298? Ans. 79993·411.

3. What is the sum of ·014, ·9816, ·32, ·15914, ·72913, and ·0047? Ans. 2·20857.

4. What is the sum of 27·148, 918·73, 14016, 294304, ·7138, and 221·7? Ans. 309488·2918.

5. Required the sum of 312·984, 21·3918, 2700·42, 3·153, 27·2, and 581·06. And. 3646·2088.

SUBTRACTION OF DECIMALS.

RULE.

1. Set the less number under the greater in the same manner as in Addition.

2. Then subtract as in whole numbers, and place the decimal point in the remainder directly under the other points.

EXAMPLES.

(1).

214·81
4·90142
209·90858

2. From ·9173 subtract ·2138. Ans. ·7035.

3. From 2·73 subtract 1·9185. Ans. 0·8115.

4. What is the difference between 91·713 and 407? Ans. 315·287.

5. What is the difference between 16·37 and 800·135? Ans. 783·715.

MULTIPLICATION OF DECIMALS.

RULE.*

1. Set down the factors under each other, and multiply them as in whole numbers.

2. And from the product, toward the right point off as many figures for decimals, as there are decimal places in both the factors. But if there be not so many figures in the product as there ought to be decimals, prefix the proper number of cyphers to supply the defect.

---

* To prove the truth of the rule, let ·9776 and ·823 be the numbers to be multiplied; now these are equivalent to $\frac{9776}{10000}$ and $\frac{823}{1000}$; whence $\frac{9776}{10000}\times\frac{823}{1000}=\frac{8045648}{10000000}=$·8045648 by the nature of Notation, and consisting of as many places, as there are cyphers, that is, of as many places as are in both the numbers; and the same is true of any two numbers whatever.

EXAMPLES.

(1)

```
   91·78
    ·381
  ------
    9178
   73424
  27534
 --------
 34·96818
```

2. What is the product of 520·3 and ·417?

Ans. 216·9651.

3. What is the product of 51·6 and 21? Ans. 1083·6.

4. What is the product of ·217 and ·0431?

Ans. ·0093527.

5. What is the product of ·051 and ·0091?

Ans. ·0004641.

NOTE. When decimals are to be multiplied by 10, or 100, or 1000, &c. that is, by 1 with any number of cyphers, it is done by only moving the decimal point so many places farther to the right, as there are cyphers in the said multiplier; subjoining cyphers, if there be not so many figures.

EXAMPLES.

1. The product of 51·3 and 10 is 513.
2. The product of 2·714 and 100 is 271·4.
3. The product of ·9163 and 1000 is 916·3.
4. The product of 21·31 and 10000 is 213100.

CONTRACTION.

When the product would contain several more decimals than are necessary for the purpose in hand, the work may be much contracted, and only the proper number of decimals retained.

RULE.

1. Set the unit figure of the multiplier under such decimal place of the multiplicand as you intend the last of your product shall be, writing the other figures of the multiplier in an inverted order.

2. Then in multiplying reject all the figures in the multiplicand, which are on the right of the figure you are multiplying by; setting down the products so that their figures on the right may fall each in a straight line under the preceding; and carrying to such figures on the right from the product of the two preceding figures in the multiplicand thus, namely, 1 from 5 to 14, 2 from 15 to 24, 3 from 25 to 34, &c. inclusively; and the sum of the lines will be the product to the number of decimals required, and will commonly be the nearest unit in the last figure.

EXAMPLES.

1. Multiply 27·14986 by 92·41035, so as to retain only four places of decimals in the product.

Contracted.	Common way.	
27·14986	27·14986	
53014·29	92·41035	
24434874	13	574930
542997	81	44958
108599	2741	986
2715	108599	44
81	542997	2
14	24434874	
2508·9280	2508·9280	650510

2. Multiply 480·14936 by 2·72416, retaining four decimals in the product.

Ans. 1308·0037.

3. Multiply 73·8429753 by 4·628754, retaining five decimals in the product. Ans. 341·80097.

4. Multiply 8634·875 by 843·7527, retaining only the integers in the product. Ans. 7285699.

DIVISION OF DECIMALS

RULE.*

Divide as in whole numbers; and to know how many decimals to point off in the quotient, observe the following rules.

---

* The reason of pointing off as many decimal places in the quotient, as those in the dividend exceed those in the divisor, will easily appear; for since the number of decimal places in the dividend is equal to those in the divisor and quotient, taken together, by the nature of Multiplication; it follows, that the quotient contains as many as the dividend exceeds the divisor.

1. There must be as many decimals in the dividend, as in both the divisor and quotient; therefore point off for decimals in the quotient so many figures, as the decimal places in the dividend exceed those in the divisor.

2. If the figures in the quotient are not so many as the rule requires, supply the defect by prefixing cyphers.

3. If the decimal places in the divisor be more than those in the dividend, add cyphers as decimals to the dividend, till the number of decimals in the dividend be equal to those in the divisor, and the quotient will be integers till all these decimals are used. And, in case of a remainder, after all the figures of the dividend are used, and more figures are wanted in the quotient, annex cyphers to the remainder, to continue the division as far as necessary.

4. The first figure of the quotient will possess the same place of integers or decimals, as that figure of the dividend, which stands over the units place of the first product.

EXAMPLES.

1. Divide 3424·6056 by 43·6.

```
43·6)3424·6056(78·546
     3052
     ────
      3726
      3488
      ────
       2380
       2180
       ────
        2005
        1744
        ────
         2616
         2616
         ────
```

2. Divide 3877875 by ·675. Ans. 5745000.
3. Divide ·0081892 by ·347. Ans. ·0236.
4. Divide 7·13 by ·18. Ans. 39.

CONTRACTIONS.

I. *If the divisor be an integer with any number of cyphers at the end;* cut them off, and remove the decimal point in the dividend so many places farther to the left, as there were cyphers cut off, prefixing cyphers, if need be; then proceed as before.

EXAMPLES.

1. Divide 953 by 21000.

```
21·000)
      3)·953
      7)·31766
        ·04538, &c.
```

Here I first divide by 3, and then by 7, because 3 times 7 is 21.

2. Divide 41020 by 32000. Ans. 1·281875.

NOTE. Hence, if the divisor be 1 with cyphers, the quotient will be the same figures with the dividend, having the decimal point so many places farther to the left, as there are cyphers in the divisor.

EXAMPLES.

217·3÷100=2·173. 419 by 10=41·9.
5·16 by 1000=·00516. ·21 by 1000=·00021.

II. *When the number of figures in the divisor is great, the operation may be contracted, and the necessary number of decimal places obtained.*

RULE.

1. Having, by the 4th general rule, found what place of decimals or integers the first figure of the quotient will possess; consider how many figures of the quotient will serve the present purpose; then take the same number of figures on the left of the divisor, and as many of the dividend figures as will contain them less than ten times; by these find the first figure of the quotient.

2. And for each following figure, divide the last remainder by the divisor, wanting one figure to the right more than before, but observing what must be carried to the first product for such omitted figures, as in the contraction of Multiplication; and continue the operation till the divisor is exhausted.

3. When there are not so many figures in the divisor, as are required to be in the quotient, begin the division with all the figures as usual, and continue it till the number of figures in the divisor and those remaining to be found in the quotient be equal; after which use the contraction.

EXAMPLES.

1. Divide 2508·928065051 by 92·41035, so as to have four decimals in the quotient.—In this case, the quotient will contain six figures. Hence

Contraction.

92·4103,5)2508·928,065051(27·1498

..... 1848207

660721 .

646872

13849 . .

9241

4608 . . .

3696

912 . . . .

832

80 . . . . .

74

6 . . . . . .

Common Way.

92·41035)2508·928065051(27·1498

1848207	0
660721	06
646872	45
13848	615
9241	035
4607	5800
3696	4140
911	16605
831	69315
79	472901
73	928280
5	544621

2. Divide 721·17562 by 2·257432, so that the quotient may contain three decimals. Ans. 319·467.

3. Divide 12·169825 by 3·14159, so that the quotient may contain five decimals. Ans. 3·87377.

4. Divide 87·076326 by 9·365407, and let the quotient contain seven decimals. Ans. 9·2976559.

REDUCTION OF DECIMALS.

CASE 1.

To reduce a vulgar fraction to its equivalent decimal.

RULE.*

Divide the numerator by the denominator, annexing as many cyphers as are necessary; and the quotient will be the decimal required.

EXAMPLES.

1. Reduce $\frac{5}{24}$ to a decimal.

$$\begin{array}{r} 4)5\cdot000000 \\ \hline 6)1\cdot250000 \\ \hline \cdot208333, \&c. \end{array}$$

2. Required the equivalent decimal expressions for $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$. Ans. ·25, ·5, and ·75.

---

* Let the vulgar fraction, whose decimal expression is required, be $\frac{7}{13}$. Now since every decimal fraction has 10, 100, 1000, &c. for its denominator; and, if two fractions be equal, it will be, as the denominator of one is to its numerator, so is the denominator of the other to its numerator; therefore 13 : 7 : : 1000, &c. : $\frac{7 \times 1000, \&c.}{13} = \frac{7000, \&c.}{13} = \cdot53846$, the numerator of the decimal required; and is the same as by the rule.

3. What is the decimal of $\frac{3}{8}$? Ans. ·375.
4. What is the decimal of $\frac{1}{25}$? Ans. ·04.
5. What is the decimal of $\frac{3}{192}$? Ans. ·015625.
6. Express $\frac{275}{3842}$ decimally. Ans. ·071577, &c.

CASE 2.

To reduce numbers of different denominations to their equivalent decimal values.

RULE.*

1. Write the given numbers perpendicularly under each other for dividends, proceeding orderly from the least to the greatest.

2. Opposite to each dividend, on the left, place such a number for a divisor, as will bring it to the next superior name, and draw a line between them.

3. Begin with the highest, and write the quotient of each division as decimal parts, on the right of the dividend next below it; and so on till they are all used, and the last quotient will be the decimal sought.

EXAMPLES.

Reduce 15s. 9¾d. to the decimal of a pound.

4	3·
12	9·75
20	15·8125

·790625 the decimal required.

* The reason of the rule may be explained from the first example; thus, three farthings are ¾ of a penny, which brought to a decimal is ·75; consequently 9¾d. may be expressed 9·75d. but 9·75 is $\frac{975}{100}$ of a penny $=\frac{975}{1200}$ of a shilling, which brought to a decimal is ·8125; and therefore 15s. 9¾d. may be expres-

2. Reduce 9s. to the decimal of a pound. Ans. ·45.

3. Reduce 19s. 5½d. to the decimal of a pound. Ans. ·972916.

4. Reduce 10oz. 18dwt. 16gr. to the decimal of a pound Troy. Ans. ·911111, &c.

5. Reduce 2qrs. 14lb. to the decimal of a cwt. Ans. ·625, &c.

6. Reduce 17yd. 1ft. 6in. to the decimal of a mile. Ans. ·00994318, &c.

7. Reduce 3qrs. 2nls. to the decimal of a yard. Ans. ·875.

8. Reduce 1gal. of wine to the decimal of a hhd. Ans. ·015873.

9. Reduce 3bu. 1pe. to the decimal of a quarter. Ans. ·40625.

10. Reduce 10w. 2d. to the decimal of a year. Ans. ·1972602, &c.

CASE 3.

To find the decimal of any number of shillings, pence, and farthings by inspection.

RULE.*

Write half the greatest even number of shillings for the first decimal figure, and let the farthings in the given pence

---

sed 15·8125s. In like manner 15·8125s. is $\frac{158125}{10000}$ of a shilling $=\frac{158125}{200000}$ of a pound =, by bringing it to a decimal, ·790625l. as by the rule.

* The invention of the rule is as follows; as shillings are so many 20ths of a pound, half of them must be so many 10ths, and consequently take the place of 10ths in the decimal; but when they are odd, their half will always consist in two figures, the first of which will be half the even number, next less, and

and farthings possess the second and third places; observing to increase the second place by 5, if the shillings be odd; and the third place by 1, when the farthings exceed 12; and by 2, when they exceed 36.

EXAMPLES.

1. Find the decimal of 15s. $8\frac{1}{2}$d. by inspection.

7	$= \frac{1}{2}$ of 14s.
5	for the odd shilling.
34	= farthings in $8\frac{1}{2}$d.
1	for the excess above 12.
·785	= decimal required.

2. Find by inspection the decimal expression of 16s. $4\frac{1}{2}$d. and 13s. $10\frac{1}{2}$d. Ans. ·819 and ·694.

3. Value the following sums by inspection, and find their total, viz. 19s. $11\frac{1}{4}$d. + 6s. 2d. + 12s. $8\frac{3}{4}$d. + 1s. $10\frac{1}{4}$d. + $\frac{3}{4}$d. + $1\frac{1}{4}$d. Ans. 2·042 the total.

---

the second as 5; and this confirms the rule as far as it respects shillings.

Again, farthings are so many 960ths of a pound; and had it happened, that 1000, instead of 960, had made a pound, it is plain any number of farthings would have made so many thousandths, and might have taken their place in the decimal accordingly. But 960, increased by $\frac{1}{24}$ part of itself, is $=1000$; consequently any number of farthings, increased by their $\frac{1}{24}$ part, will be an exact decimal expression for them. Whence, if the number of farthings be more than 12, a $\frac{1}{24}$ part is greater than $\frac{1}{2}$, and therefore 1 must be added; and when the number of farthings is more than 36, a $\frac{1}{24}$ part is greater than $1\frac{1}{2}$, for which 2 must be added; and thus the rule is shown to be right.

CASE 4.

To find the value of any given decimal in terms of the integer.

RULE.

1. Multiply the decimal by the number of parts in the next less denomination, and cut off as many places for a remainder on the right as there are places in the given decimal.

2. Multiply the remainder by the parts in the next inferior denomination, and cut off for a remainder as before.

3. Proceed in this manner through all the parts of the integer, and the several denominations, standing on the left, make the answer.

EXAMPLES.

1. Find the value of ·37623 of a pound.

```
    20
-------
7·52460
     12
-------
6·29520
      4
-------
1·18080      Ans. 7s. 6¼d.
```

2. What is the value of ·625 of a shilling? Ans. 7½d.

3. What is the value of ·8322916l.? Ans. 16s. 7½d.

4. What is the value of ·6725cwt.? Ans. 2qrs. 19lb. 5oz.

5. What is the value of ·67 of a league?
Ans. 2mls. 3pls. 1yd. 3in. 1b. c.

6. What is the value of ·61 of a tun of wine?
Ans. 2hhd. 27gal. 2qt. 1pt.

7. What is the value of ·461 of a chaldron of coals?
Ans. 16bu. 2pe.

8. What is the value of ·42857 of a month?
Ans. 1w. 4d. 23h. 59′ 56″.

CASE 5.

To find the value of any decimal of a pound by inspection.

RULE.

Double the first figure or place of tenths for shillings, and if the second be 5 or more than 5 reckon another shilling; then call the figures in the second and third places, after 5, if contained, is deducted, so many farthings; abating 1, when they are above twelve; and 2, when above 36; and the result is the answer.

EXAMPLES.

1. Find the value of ·785l. by inspection.

14s. = double 7.
1s. for 5 in the place of tenths.
$8\frac{3}{4}$ = 35 farthings.
$\frac{1}{4}$ for the excess of 12, abated.

15s. $8\frac{1}{2}$d. the answer.

2. Find the value of ·875l. by inspection. Ans. 17s. 6d.

3. Value the following decimals by inspection, and find their sum, viz. ·927 + ·351l. + ·203l. + ·061l. + 02l. + ·009l. Ans. 1l. 11s. $5\frac{1}{4}$d.

FEDERAL MONEY.*

The denominations of *Federal Money*, as determined by an Act of Congress, Aug. 8, 1786, are in a *decimal* ratio; and therefore may be properly introduced in this place.

* The coins of federal money are two of gold, four of silver, and two of copper. The gold coins are called an *eagle* and *half-eagle*; the silver, a *dollar*, *half-dollar*, *double-dime*, and *dime*; and the copper, a *cent* and *half-cent*. The standard for gold

A mill, which is the lowest money of account, ·001 of a dollar, which is the money unit.

A cent	is	·01	Or 10 mills = 1 cent.	
A dime		·1	*marked*	m. c.
A dollar		1·	10 cents = 1 dime,	d.
An eagle		10·	10 dimes = 1 dollar,	D.
			10 dollars = eagle,	E.

---

and silver is eleven parts fine and one part alloy. The weight of fine gold in the eagle is 246·268 grains; of fine silver in the dollar, 375·64 grains; of copper in 100 cents, $2\frac{1}{4}$lb. Avoirdupois. The fine gold in the half-eagle is half the weight of that in the eagle; the fine silver in the half-dollar, half the weight of that in the dollar, &c. The denominations less than a dollar are expressive of their values: thus, *mill* is an abbreviation of *mille*, a thousand, for 1000 mills are equal to 1 dollar; *cent*, of *centum*, a hundred, for 100 cents are equal to 1 dollar; a *dime* is the French of *tithe*, the tenth part, for 10 dimes are equal to 1 dollar.

The mint-price of uncoined gold, 11 parts being fine and 1 part alloy, is 209 dollars, 7 dimes, and 7 cents per lb. Troy weight; and the mint-price of uncoined silver, 11 parts being fine and 1 part alloy, is 9 dollars, 9 dimes, and 2 cents, per lb. Troy.

In Mr. PIKE's "Complete System of Arithmetic," may be seen "RULES for reducing the Federal Coin, and the Currencies of the several United States; also English, Irish, Canada, Nova Scotia, Livres Tournois, and Spanish milled dollars, each to the *par* of all the others." It may be sufficient here to observe respecting the currencies of the several States, that a dollar is equal to 6s. in New-England and Virginia; 8s. in New-York and North-Carolina; 7s. 6d. in New-Jersey, Pennsylvania, Delaware, and Maryland; and 4s. 8d. in South-Carolina and Georgia.

The English standard for gold is 22 carats of fine gold, and

A number of dollars, as 754, may be read 754 dollars, or 75 eagles, 4 dollars; and decimal parts of a dollar, as ·365, may be read 3 dimes, 6 cents, 5 mills, or 36 cents, 5 mills, or 365 mills; and others in a similar manner.

Addition, *Subtraction*, *Multiplication*, and *Division* of federal money are performed just as in decimal fractions; and consequently with more ease than in any other kind of money.

EXAMPLES.

1. Add 2 dollars, 4 dimes, 6 cents, 4D. 2d., 4d., 9c., 1E. 3D. 5c. 7m., 3c. 9m., 1D. 2d. 8c. 1m., and 2E. 4D. 7d. 8c. 2m. together.

(1) E. D. d. c. m.	(2) E. D. d. c. m.	(3) E. D. d. c. m.
2·46	34·123	30·671
4·2	1·178	3·123
·39	78·001	4·567
13·057	1·7	·03
·039	·32	70·308
1·281	61·789	7·17
24·782	6·341	8·231
46·309 Ans.		

---

2 carats of copper, which is the same as 11 parts fine and 1 part alloy. The English standard for silver is 18oz. 2dwt. of fine silver, and 18dwt. of copper; so that the proportion of alloy in their silver is less than in their gold. When either gold or silver is finer or coarser than standard, the variation from standard is estimated by carats and grains of a carat in gold, and by penny-weights in silver. Alloy is used in gold and silver to harden them.

NOTE.—Carat is not any certain weight or quantity, but $\frac{1}{24}$ of any weight or quantity; and the minters and goldsmiths divide it into 4 equal parts, called *grains* of a carat.

	(4)	(5)	(6)
	E. D. d. c. m.	E. D. d. c. m.	D. d. c. m.
From	3 2 · 1 7 8	7 0 · 0 0 0	2 · 6 5 2
Subtract	1 7 · 2 8 9	7 · 8 1 3	· 0 7
Remain.	1 4 · 8 8 9		

7. Multiply 3D. 4d. 5c. 1m. by 1D. 2d. 3c. 2m.

```
   D.
   3·451
   1·232
  ------
   6902
  10353
  6902
 3451
-------
```

4·251632=4·251$\frac{632}{1000}$ Ans.

NOTE. The figures after or on the right of mills are decimals of a mill.

8. Multiply 6.347 D. by 4·532 D. Ans. 28·764604 D.

9. Multiply 71·012 D. by 3·703 D. Ans. 262·957436 D.

10. Multiply 806·222$\frac{2}{9}$ D by 9 Ans. 7256 D.

11. Divide 4·251632 D. by 1·232 D.

```
1·232)4.251632(3·451 Answer.
      3696
      ----
       5556
       4928
       ----
        6283
        6160
        ----
         1232
         1232
         ----
```

12. Divide 20D. by 2000. Ans. D. 0·01.

13. Divide 7256D. by 9. Ans. D. 806·222$\frac{2}{9}$.

CIRCULATING DECIMALS.

It has already been observed, that when an infinite decimal repeats always one figure, it is a *single repetend;* and when more than one, a *compound repetend;* also that a point is set over a single repetend, and a point over the first and last figures of a compound repetend.

It may be farther observed, that when other decimal figures precede a repetend in any number, it is called a *mixed repetend:* as $\cdot2\dot{3}$, or $\cdot104\dot{1}2\dot{3}$; otherwise it is a *pure*, or *simple*, *repetend:* as $\cdot\dot{3}$ and $\cdot\dot{1}2\dot{3}$.

Similar repetends begin at the same place: as $\cdot\dot{3}$ and $\cdot\dot{6}$, or $1\cdot\dot{3}4\dot{1}$ and $2\cdot\dot{1}5\dot{6}$.

Dissimilar repetends begin at different places: as $\cdot\dot{2}5\dot{3}$ and $\cdot47\dot{5}\dot{2}$.

Conterminous repetends end at the same place: as $\cdot\dot{1}2\dot{5}$ and $\cdot\dot{0}0\dot{9}$.

Similar and *conterminous repetends* begin and end at the same place: as $2\cdot9\dot{1}0\dot{4}$ and $\cdot0\dot{6}1\dot{3}$.

REDUCTION OF CIRCULATING DECIMALS.

CASE 1.

To reduce a simple repetend to its equivalent vulgar fraction.

RULE.*

1. Make the given decimal the numerator, and let the denominator be a number, consisting of as many nines as there are recurring places in the repetend.

2. If there be integral figures in the circulate, as many cyphers must be annexed to the numerator, as the highest place of the repetend is distant from the decimal point.

EXAMPLES.

1. Required the least vulgar fractions equal to $\cdot\dot{6}$ and $\cdot\dot{1}2\dot{3}$.

$\cdot\dot{6}=\frac{6}{9}=\frac{2}{3}$; and $\cdot\dot{1}2\dot{3}=\frac{123}{999}=\frac{41}{333}$ Ans.

2. Reduce $\cdot\dot{3}$ to its equivalent vulgar fraction. Ans. $\frac{1}{3}$.

---

* If unity, with cyphers annexed, be divided by 9 *ad infinitum*, the quotient will be 1 continually; i. e. if $\frac{1}{9}$ be reduced to a decimal, it will produce the circulate $\cdot\dot{1}$; and since $\cdot\dot{1}$ is the decimal equivalent to $\frac{1}{9}$, $\cdot\dot{2}$ will $=\frac{2}{9}$, $\cdot\dot{3}=\frac{3}{9}$, and so on till $\cdot\dot{9}=\frac{9}{9}=1$.

Therefore every single repetend is equal to a vulgar fraction, whose numerator is the repeating figure and denominator 9.

Again, $\frac{1}{99}$, or $\frac{1}{999}$, being reduced to decimals, makes $\cdot010101$, &c. or $\cdot001001$, &c. *ad infinitum* $=\cdot\dot{0}\dot{1}$ or $\cdot\dot{0}0\dot{1}$; that is, $\frac{1}{99}=\cdot\dot{0}\dot{1}$, and $\frac{1}{999}=\cdot\dot{0}0\dot{1}$; consequently $\frac{2}{99}=\cdot\dot{0}\dot{2}$, $\frac{3}{99}=\cdot\dot{0}\dot{3}$, &c. and $\frac{2}{999}=\cdot\dot{0}0\dot{2}$, $\frac{3}{999}=\cdot\dot{0}0\dot{3}$, &c. and the same will hold universally.

3. Reduce 1·62 to its equivalent vulgar fraction.

Ans. $\frac{1620}{999}$.

4. Required the least vulgar fraction equal to $\cdot\dot{7}6923\dot{0}$.

Ans. $\frac{10}{13}$.

CASE 2.

To reduce a mixed repetend to its equivalent vulgar fraction.

RULE.*

1. To as many nines as there are figures in the repetend, annex as many cyphers as there are finite places, for a denominator.

2. Multiply the nines in the said denominator by the finite part, and add the repeating decimal to the product, for the numerator.

3. If the repetend begin in some integral place, the finite value of the circulating part must be added to the finite part.

EXAMPLES.

1. What is the vulgar fraction equivalent to $\cdot 13\dot{8}$?

$9\times 13+8=125=$ numerator, and $900=$ the denominator; $\therefore$ $\cdot 13\dot{8}=\frac{125}{900}=\frac{5}{36}$ the answer.

---

* In like manner for a mixed circulate; consider it as divisible into its finite and circulating parts, and the same principle will be seen to run through them also: thus, the mixed circulate $\cdot 1\dot{6}$ is divisible into the finite decimal $\cdot 1$, and the repetend $\cdot 0\dot{6}$; but $\cdot 1=\frac{1}{10}$, and $\cdot 0\dot{6}$ would be $=\frac{6}{9}$, provided the circulation began immediately after the place of units; but as it begins after the place of tens, it is $\frac{6}{9}$ of $\frac{1}{10}=\frac{6}{90}$, and so the vulgar fraction $=\cdot 1\dot{6}$ is $\frac{1}{10}+\frac{6}{90}=\frac{9}{90}+\frac{6}{90}=\frac{15}{90}$, and is the same as by the rule.

2. What is the least vulgar fraction equivalent to $\cdot 5\dot{3}$?

Ans. $\frac{8}{15}$.

3. What is the least vulgar fraction equal to $\cdot 5\dot{9}2\dot{5}$?

Ans. $\frac{16}{27}$.

4. What is the least vulgar fraction equal to $\cdot 008\dot{4}9713\dot{3}$?

Ans. $\frac{83}{9768}$.

5. What is the finite number equivalent to $3\dot{1}\cdot 6\dot{2}$?

Ans. $31\frac{23}{37}$.

CASE 3.

To make any number of dissimilar repetends similar and conterminous.

RULE.*

Change them into other repetends, which shall each consist of as many figures as the least common multiple of the several numbers of places, found in all the repetends, contains units.

EXAMPLES.

1.	Dissimilar.		Made similar and conterminous.
	$9\cdot\dot{8}1\dot{4}$	=	$9\cdot 81\dot{4}8148\dot{1}$
	$1\cdot 5$	=	$1\cdot 5\dot{0}00000\dot{0}$
	$87\cdot 2\dot{6}$	=	$87\cdot 2\dot{6}66666\dot{6}$
	$\cdot 08\dot{4}$	=	$\cdot 08\dot{3}33333\dot{3}$
	$124\cdot\dot{0}\dot{9}$	=	$124\cdot 0\dot{9}09090\dot{9}$

* Any given repetend whatever, whether single, compound, pure, or mixed, may be transformed into another repetend, that shall consist of an equal or greater number of figures at pleasure: thus $\cdot\dot{4}$ may be transformed to $\cdot\dot{4}\dot{4}$, or $\cdot\dot{4}4\dot{4}$, or $\cdot 4\dot{4}$, &c. Also $\cdot\dot{5}\dot{7} = \cdot\dot{5}75\dot{7} = \cdot 57\dot{5}\dot{7} = \cdot 5\dot{7}\dot{5}$; and so on; which is too evident to need any further demonstration.

2. Make $\cdot\dot{3}$, $\cdot2\dot{7}$ and $\cdot04\dot{5}$ similar and conterminous.

3. Make $\cdot\dot{3}2\dot{1}$, $\cdot82\dot{6}\dot{2}$, $\cdot\dot{0}\dot{5}$ and $\cdot0\dot{9}0\dot{2}$ similar and conterminous.

4. Make $\cdot521\dot{7}$, $3\cdot\dot{6}4\dot{3}$ and $17\cdot\dot{1}2\dot{3}$ similar and conterminous.

CASE 4.

To find whether the decimal fraction, equal to a given vulgar one, be finite or infinite, and of how many places the repetend will consist.

RULE.*

1. Reduce the given fraction to its least terms, and divide the denominator by 2, 5, or 10, as often as possible.

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* In dividing 1·0000, &c. by any prime number whatever, except 2 or 5, the figures in the quotient will begin to repeat as soon as the remainder is 1. And since 9999, &c. is less than 10000, &c. by 1, therefore 9999, &c. divided by any number whatever will leave 0 for a remainder, when the repeating figures are at their period. Now whatever number of repeating figures we have, when the dividend is 1, there will be exactly the same number, when the dividend is any other number whatever. For the product of any circulating number, by any other given number, will consist of the same number of repeating figures as before. Thus, let ·507650765076, &c. be a circulate, whose repeating part is 5076. Now every repetend (5076) being equally multiplied, must produce the same product. For though these products will consist of more places, yet the overplus in each, being alike, will be carried to the next, by which means each product will be equally increased, and consequently every four places will continue alike. And the same will hold for any other number whatever.

2. If the whole denominator vanish in dividing by 2, 5, or 10, the decimal will be finite, and will consist of so many places, as you perform divisions.

3. If it do not so vanish, divide 9999, &c. by the result, till nothing remain, and the number of 9s used will show the number of places in the repetend; which will begin after so many places of figures, as there were 10s, 2s, or 5s, used in dividing.

EXAMPLES.

1. Required to find whether the decimal equal to $\frac{210}{1120}$ be finite or infinite; and if infinite, how many places the repetend will consist of.

$\frac{210}{1120}=\frac{3}{16}\ \overset{2}{|}\ 8\ \overset{2}{|}\ 4\ \overset{2}{|}\ 2\ |\ 1$; therefore the decimal is finite, and consists of 4 places.

2. Let $\frac{1}{11}$ be the fraction proposed.

3. Let $\frac{2}{7}$ be the fraction proposed.

4. Let $\frac{13}{404}$ be the fraction proposed.

5. Let $\frac{1}{3544}$ be the fraction proposed.

ADDITION OF CIRCULATING DECIMALS.

RULE.*

1. Make the repetends similar and conterminous, and find their sum as in common Addition.

Now hence it appears, that the dividend may be altered at pleasure, and the number of places in the repetend will still be the same: thus $\frac{1}{11}=\cdot\dot{0}\dot{9}$, and $\frac{3}{11}$, or $\frac{1}{11}\times 3=\cdot\dot{2}\dot{7}$, where the number of places in each is alike, and the same will be true in all cases.

* These rules are both evident from what has been said in reduction.

2. Divide this sum by as many nines as there are places in the repetend, and the remainder is the repetend of the sum; which must be set under the figures added, with cyphers on the left, when it has not so many places as the repetends.

3. Carry the quotient of this division to the next column, and proceed with the rest as in finite decimals.

EXAMPLES.

1. Let $3{\cdot}\dot{6}+78{\cdot}3\dot{4}7\dot{6}+735{\cdot}\dot{3}+375+{\cdot}\dot{2}\dot{7}+187{\cdot}\dot{4}$ be added together.

Dissimilar.		Similar and conterminous.
$3{\cdot}\dot{6}$	=	$3{\cdot}6\dot{6}6666\dot{6}$
$78{\cdot}3\dot{4}7\dot{6}$	=	$78{\cdot}3\dot{4}7647\dot{6}$
$735{\cdot}\dot{3}$	=	$735{\cdot}3\dot{3}3333\dot{3}$
$375{\cdot}$	=	$375{\cdot}0\dot{0}0000\dot{0}$
${\cdot}\dot{2}\dot{7}$	=	$0{\cdot}2\dot{7}2727\dot{2}$
$187{\cdot}\dot{4}$	=	$187{\cdot}4\dot{4}4444\dot{4}$

$1380{\cdot}0\dot{6}4819\dot{3}$ the sum.

In this question, the sum of the repetends is 2648191, which, divided by 999999, gives 2 to carry, and the remainder is 648193.

2. Let $5391{\cdot}357+72{\cdot}3\dot{8}+187{\cdot}2\dot{1}+4{\cdot}296\dot{5}+217{\cdot}849\dot{6}+42{\cdot}17\dot{6}+{\cdot}52\dot{3}+58{\cdot}3004\dot{8}$ be added together.

Ans. $5974{\cdot}1037\dot{1}$.

3. Add $9{\cdot}\dot{8}1\dot{4}+1{\cdot}5+87{\cdot}2\dot{6}+{\cdot}08\dot{3}+124{\cdot}\dot{0}\dot{9}$ together.

Ans. $222{\cdot}7\dot{5}57239\dot{0}$.

4. Add $\dot{1}6\dot{2}+134\cdot\dot{0}\dot{9}+2\cdot\dot{9}\dot{3}+97\cdot2\dot{6}+3\cdot\dot{7}6923\dot{0}+99\cdot08\dot{3}+1\cdot5+\cdot\dot{8}1\dot{4}$ together. Ans. $501\cdot62\dot{6}5107\dot{7}$.

SUBTRACTION OF CIRCULATING DECIMALS.

RULE.

Make the repetends similar and conterminous, and subtract as usual; observing, that, if the repetend of the subtrahend be greater than the repetend of the minuend, then the figure of the remainder on the right must be less by unity, than it would be, if the expressions were finite.

EXAMPLES.

1. From $85\cdot\dot{6}\dot{2}$ take $13\cdot\dot{7}643\dot{2}$.

$$85\cdot\dot{6}\dot{2} = 85\cdot62\dot{6}2\dot{6}$$
$$13\cdot\dot{7}643\dot{2} = 13\cdot7\dot{6}43\dot{2}$$

$71\cdot8\dot{6}19\dot{3}$ the difference.

2. From $476\cdot3\dot{2}$ take $84\cdot769\dot{7}$. Ans. $391\cdot552\dot{4}$.

3. From $3\cdot85\dot{6}\dot{4}$ take $\cdot03\dot{8}\dot{2}$. Ans. $3\cdot8\dot{1}$.

MULTIPLICATION OF CIRCULATING DECIMALS.

RULE.

1. Turn both the terms into their equivalent vulgar fractions, and find the product of those fractions as usual.

2. Turn the vulgar fraction, expressing the product, into an equivalent decimal, and it will be the product required.

EXAMPLES.

1. Multiply $\cdot\dot{3}\dot{6}$ by $\cdot 2\dot{5}$.

$$\cdot\dot{3}\dot{6}=\tfrac{36}{99}=\tfrac{4}{11}$$

$$\cdot 2\dot{5}=\tfrac{23}{90}$$

$\tfrac{4}{11}\times\tfrac{23}{90}=\tfrac{92}{990}=\cdot 09\dot{2}\dot{9}$ the product.

2. Multiply $37\cdot 2\dot{3}$ by $\cdot 2\dot{6}$. Ans. $9\cdot 92\dot{8}$.

3. Multiply $8574\cdot\dot{3}$ by $87\cdot\dot{5}$. Ans. $750730\cdot 5\dot{1}\dot{8}$.

4. Multiply $3\cdot\dot{9}7\dot{3}$ by 8. Ans. $31\cdot\dot{7}9\dot{1}$.

5. Multiply $49640\cdot 5\dot{4}$ by $\cdot 7050\dot{3}$. Ans. $34998\cdot 41\dot{9}900\dot{3}$.

6. Multiply $3\cdot 1\dot{4}\dot{5}$ by $4\cdot\dot{2}9\dot{7}$. Ans. $13\cdot 5169\dot{5}3\dot{3}$.

DIVISION OF CIRCULATING DECIMALS.

RULE.

1. Change both the divisor and dividend into their equivalent vulgar fractions, and find their quotient as usual.

2. Turn the vulgar fraction, expressing the quotient, into its equivalent decimal, and it will be the quotient required.

EXAMPLES.

1. Divide $\cdot\dot{3}\dot{6}$ by $\cdot 2\dot{5}$.

$$\cdot\dot{3}\dot{6}=\tfrac{36}{99}=\tfrac{4}{11}$$

$$\cdot 2\dot{5}=\tfrac{23}{90}$$

$\tfrac{4}{11}\div\tfrac{23}{90}=\tfrac{4}{11}\times\tfrac{90}{23}=\tfrac{360}{253}=1\tfrac{107}{253}=1\cdot\dot{4}22924901185770750988\dot{1}$ the quotient.

2. Divide $319\cdot 280071\dot{1}\dot{2}$ by $764\cdot\dot{5}$. Ans. $\cdot 41763\dot{2}\dot{5}$.

3. Divide $234\cdot\dot{6}$ by $\cdot\dot{7}$. Ans. $301\cdot\dot{7}1428\dot{5}$.

4. Divide $13\cdot 5169\dot{5}3\dot{3}$ by $4\cdot\dot{2}9\dot{7}$. Ans. $3\cdot 1\dot{4}\dot{5}$.

PROPORTION IN GENERAL.

NUMBERS are compared together to discover the relations they have to each other.

There must be two numbers to form a comparison; the number, which is compared, being written first, is called the *antecedent;* and that, to which it is compared, the *consequent.* Thus of these numbers, 2 : 4 : : 3 : 6, 2 and 3 are called the antecedents; and 4 and 6, the consequents.

Numbers are compared to each other two different ways; one comparison considers the *difference* of the two numbers, and is called *arithmetical relation*, the difference being sometimes named the *arithmetical ratio;* and the other considers their *quotient*, and is termed *geometrical relation*, and the quotient the *geometrical ratio.* So of these numbers 6 and 3, the difference or arithmetical ratio is 6—3 or 3; and the geometrical ratio is $\frac{6}{3}$ or 2.

If two or more couplets of numbers have equal ratios, or differences, the equality is named *proportion;* and their terms similarly posited, that is, either all the greater, or all the less, taken as antecedents, and the rest as consequents, are called *proportionals.* So the two couplets 2, 4, and 6, 8, taken thus, 2, 4, 6, 8, or thus 4, 2, 8, 6, are arithmetical proportionals; and the couplets 2, 4, and 8, 16, taken thus, 2, 4, 8, 16, or thus, 4, 2, 16, 8, are geometrical proportionals.*

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* In geometrical proportionals a colon is placed between the terms of each couplet, and a double colon between the couplets; in arithmetical proportionals a colon may be turned horizontally between the terms of each couplet, and two colons written between the couplets. Thus the above geometrical proportionals are written thus, 2 : 4 : : 8 : 16, and 4 : 2 : : 16 : 8; the arithmetical, 2 ·· 4 :: 6 ·· 8, and 4 ·· 2 : : 8 ·· 6.

Proportion is distinguished into *continued* and *discontinued.* If, of several couplets of proportionals written in a series, the difference or ratio of each consequent and the antecedent of the next following couplet be the same as the common difference or ratio of the couplets, the proportion is said to be *continued*, and the numbers themselves a series of *continued arithmetical* or *geometrical proportionals.* So 2, 4, 6, 8, form an arithmetical progression; for 4—2=6—4=8—6=2; and 2, 4, 8, 16, a geometrical progression; for $\frac{4}{2}=\frac{8}{4}=\frac{16}{8}=2$.

But if the difference or ratio of the consequent of one couplet and the antecedent of the next couplet be not the same as the common difference or ratio of the couplets, the proportion is said to be *discontinued.* So 4, 2, 8, 6, are in *discontinued arithmetical proportion;* for 4—2=8—6=2, but 8—2=6; also 4, 2, 16, 8, are in *discontinued geometrical proportion;* for $\frac{4}{2}=\frac{16}{8}=2$, but $\frac{16}{2}=8$.

Four numbers are *directly proportional*, when the ratio of the first to the second is the same, as that of the third to the fourth. As 2 : 4 : : 3 : 6. Four numbers are said to be *reciprocally*, or *inversely proportional*, when the first is to the second, as the fourth is to the third, and vice versa. Thus, 2, 6, 9, and 3, are reciprocal proportionals; 2 : 6 : : 3 : 9.

Three or four numbers are said to be in *harmonical proportion*, when, in the former case, the difference of the first and second is to the difference of the second and third, as the first is to the third; and, in the latter, when the difference of the first and second is to the difference of the third and fourth, as the first is to the fourth. Thus, 2, 3, and 6; and 3, 4, 6, and 9, are harmonical proportionals; for 3—2=1 : 6—3=3 : : 2 : 6; and 4—3=1 : 9—6=3 : : 3 : 9.

Of four arithmetical proportionals the sum of the extremes is equal to the sum of the means.* Thus of 2 ·· 4 : : 6 ·· 8 the sum of the extremes (2+8)= the sum of the means (4+6)=10. Therefore, of three arithmetical proportionals, the sum of the extremes is double the mean.

Of four geometrical proportionals, the product of the extremes is equal to the product of the means.† Thus, of 2 : 4 : : 8 : 16, the product of the extremes (2×16) is equal to the product of the means (4×8)=32. Therefore of three geometrical proportionals, the product of the extremes is equal to the square of the mean.

Hence it is easily seen, that either extreme of four geometrical proportionals is equal to the product of the means divided by the other extreme; and that either mean is equal to the product of the extremes divided by the other mean.

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* Demonstration. Let the four arithmetical proportionals be A, B, C, D, viz. $A \cdot\cdot B :: C \cdot\cdot D$; then, $A-B=C-D$, and $B+D$ being added to both sides of the equation, $A-B+B+D=C-D+B+D$; that is, $A+D$ the sum of the extremes $=C+B$ the sum of the means.—And three A, B, C, may be thus expressed, $A \cdot\cdot B :: B \cdot\cdot C$; therefore $A+C=B+B=2B$. Q. E. D.

† Demonstration. Let the proportion be $A : B :: C : D$, and let $\frac{A}{B}=\frac{C}{D}=r$; then $A=Br$, and $C=Dr$; multiply the former of these equations by D, and the latter by B; then $AD=BrD$, and $CB=DrB$, and consequently AD the product of the extremes is equal to BC the product of the means.—And three may be thus expressed, $A : B :: B : C$, therefore $AC=B\times B=B^2$. Q. E. D.

SIMPLE PROPORTION, OR RULE OF THREE.

The Rule of Three is that, by which a number is found, having to a given number the same ratio, which is between two other given numbers. For this reason it is sometimes named the *Rule of Proportion*.

It is called the *Rule of Three*, because in each of its questions there are given *three numbers* at least. And because of its excellent and extensive use, it is often named the *Golden Rule*.

RULE.*

1. Write the number, which is of the same kind with the answer or number required.

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* DEMONSTRATION. The following observations taken collectively, form a demonstration of the rule, and of the reductions mentioned in the notes subsequent to it.

1. There can be comparison or ratio between two numbers, only when they are considered abstractly, or as applied to things of the same kind, so that one can, in a proper sense, be contained in the other. Thus there can be no comparison between 2 men and 4 days; but there may be between 2 and 4, and between 2 days and 4 days, or 2 men and 4 men. Therefore, the 2 of the 3 given numbers, that are of the same kind, that is, the first and the third, when they are stated according to the rule, are to be compared together, and their ratio is equal to that, required between the remaining or second number and the fourth or answer.

2. Though numbers of the same kind, being either of the same or of different denominations, have a real ratio, yet this ratio is the same as that of the two numbers taken abstractly, only when they are of the same denomination. Thus the ratio of 1l. to 2l. is the same as that of 1 to 2 $=\frac{1}{2}$; 1s. has a real ratio to 2l. but it is not the ratio of 1 to 2; it is the ratio of 1s.

2. Consider whether the answer ought to be greater or less than this number; if greater, write the greater of the

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to 40s. that is, of 1 to 40 $=\frac{1}{40}$. Therefore, as the first and third numbers have the ratio, that is required between the second and answer, they must, if not of the same denomination, be reduced to it; and then their ratio is that of the abstract numbers.

3. The product of the extremes of four geometrical proportionals is equal to the product of the means; hence, if the product of two numbers be equal to the product of two other numbers, the four numbers are proportionals; and if the product of two numbers be divided by a third, the quotient will be a fourth proportional to those three numbers. Now as the question is resolvable into this, viz. to find a number of the same kind as the second in the statement, and having the same ratio to it, that the greater of the other two has to the less, or the less has to the greater; and as these two, being of the same denomination, may be considered as abstract numbers; it plainly follows, that the fourth number or answer is truly found by multiplying the second by one of the other two, and dividing the product by that which remains.

4. It is very evident, that, if the answer must be greater than the second number, the greater of the other two numbers must be the multiplier, and may occupy the third place; but, if less, the less number must be the multiplier.

5. The reduction of the second number is only performed for convenience in the subsequent multiplication and division, and not to produce an abstract number. The reason of the reduction of the quotient, of the remainder after division, and of the product of the second and third terms, when it cannot be divided by the first is obvious.

6. If the second and third numbers be multiplied together, and the product be divided by the first; it is evident, that the

two remaining numbers on the right of it for the third, and the other on the left for the first number or term; but if less

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answer remains the same, whether the number compared with the first be in the second or third place.

Thus is the proposed demonstration completed.

There are four other methods of operation beside the general one given above, any of which, when applicable, performs the work much more concisely. They are these:

1. Divide the second term by the first, multiply the quotient by the third, and the product will be the answer.

2. Divide the third term by the first, multiply the quotient by the second, and the product will be the answer.

3. Divide the first term by the second, divide the third by the quotient, and the last quotient will be the answer.

4. Divide the first term by the third, divide the second by the quotient, and the last quotient will be the answer.

The general rule above given is equivalent to those, which are usually given in the direct and inverse rules of three, and which are here subjoined.

The RULE OF THREE DIRECT teaches, by having three numbers given, to find a fourth, that shall have the same proportion to the third, as the second has to the first.

RULE.

1. State the question; that is, place the numbers so, that the first and third may be the terms of supposition and demand, the second of the same kind with the answer required.

2. Bring the first and third numbers into the same denomination, and the second into the lowest name mentioned.

3. Multiply the second and third numbers together, and divide the product by the first, and the quotient will be the answer to the question, in the same denomination you left the second number in; which may be brought into any other denomination required.

write the less of the two remaining numbers in the third place, and the other in the first.

EXAMPLE.

If 24lb. of raisins cost 6s. 6d. what will 18 frails cost, each weighing net 3qrs. 18lb.?

```
24lb. : 6s. 6d. : : 18 frails, each 3qrs. 18lb. :
          12                      28
         ---                    -----
          78                     102
                                  18
                                -----
                                 816
                                102
                                -----
                                1836
                                  78
                                -----
                               14688
                              12852
                              ------   12)
                           24)143208 (   5967
                               232      -----
                                160    2,0)49,7  3
                                168     -----
         Ans. 24l. 17s. 3d.   ------  £.24  17  3
```

The rule is founded on this obvious principle, that the magnitude or quantity of any effect varies constantly in proportion to the varying part of the cause: thus the quantity of goods bought is in proportion to the money laid out; the space gone over by an uniform motion is in proportion to the time, &c. The truth of the rule, as applied to ordinary inquiries, may be made very evident by attending only to the principles of Compound Multiplication and Division. It is shown in Multiplication of money, that the price of one, multiplied by the quantity, is the price of the whole; and in Division, that the price of

3. Multiply the second and third terms together, divide the product by the first, and the quotient will be the answer.

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the whole, divided by the quantity is the price of one. Now, in all cases of valuing goods, &c. where one is the first term of the proportion, it is plain, that the answer, found by this rule, will be the same as that found by Multiplication of money; and where one is the last term of the proportion, it will be the same as that found by the Division of money. In like manner, if the first term be any number whatever, it is plain, that the product of the second and third terms will be greater than the true answer required by as much as the price in the second term exceeds the price of one, or as the first term exceeds an unit. Consequently this product divided by the first term will give the true answer required, and is the rule.

There will sometimes be difficulty in separating the parts of complicated questions, where two or more statings are required, and in preparing the questions for stating, or after a proportion is wrought; but as there can be no general directions given for the management of these cases, it must be left to the judgment and experience of the learner.

The RULE OF THREE INVERSE teaches, by having three numbers given to find a fourth, that shall have the same proportion to the second, as the first has to the third.

If *more* require *more*, or *less* require *less*, the question belongs to the Rule of Three Direct.

But if *more* require *less*, or *less* require *more*, it belongs to the Rule of Three Inverse.

NOTE. The meaning of these phrases, "if *more* require *more*, *less* require *less*," &c. is to be understood thus: *more* requires *more*, when the third term is greater than the first, and requires the fourth to be greater than the second; *more* requires *less*, when the third term is greater than the first, and requires the fourth to be less than the second; *less* requires *more*, when the

NOTE 1. It is sometimes most convenient to multiply and divide as in Compound Multiplication and Division;

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third term is less than the first, and requires the fourth to be greater than the second; and *less* requires *less*, when the third term is less than the first, and requires the fourth to be less than the second.

RULE.

1. State and reduce the terms as in the rule of three direct.

2. Multiply the first and second terms together, and divide their product by the third, and the quotient is the answer to the question, in the same denomination you left the second number in.

The method of proof, whether the proportion be direct or inverse, is by inverting the question.

EXAMPLE.

What quantity of shalloon, that is three quarters of a yard wide, will line $7\frac{1}{2}$ yards of cloth, that is $1\frac{1}{2}$ yard wide?

```
lyd. 2qrs. : 7yds. 2qrs. : : 3qrs. :
 4            4
 —           ——
 6           30
              6
           ————
           3)180
           ————
            4)60
            ——
```

15 yards, the answer.

The reason of this rule may be explained from the principles of Compound Multiplication and Division, in the same manner as the direct rule. *For example;* If 6 men can do a piece of work in 10 days, in how many days will 12 men do it?

As 6 *men* : 10 *days* : : 12 *men*: $\frac{6\times10}{12}$=5 *days*, the answer.

and sometimes it is expedient to multiply and divide according to the rules of vulgar or decimal fractions. But when neither of these modes is adopted, reduce the compound terms, each to the lowest denomination mentioned in it, and the first and third to the same denomination; then will the answer be of the same denomination with the second term. And the answer may afterward be brought to any denomination required.

NOTE 2. When there is a remainder after division, reduce it to the denomination next below the last quotient, and divide by the same divisor, so shall the quotient be so many of the said next denomination; proceed thus, as long as there is any remainder, till it is reduced to the lowest denomination, and all the quotients together will be the answer. And when the product of the second and third terms cannot be divided by the first, consider that product as a remainder after division, and proceed to reduce and divide it in the same manner.

NOTE 3. If the first term and either the second or third can be divided by any number without a remainder, let them be divided, and the quotient used instead of them.

Direct and *inverse* proportion are properly only parts of the same general rule, and are both included in the preceding.

Two or more statings are sometimes necessary, which may always be known from the nature of the question.

The method of proof is by inverting the question.

And here the product of the first and second terms, that is, 6 times 10, or 60, is evidently the time, in which one man would perform the work; therefore 12 men will do it in one twelfth part of that time, or 5 days; and this reasoning is applicable to any other instance whatever.

EXAMPLES.

1. Let it be proposed to find the value of 14oz. 8dwt. of gold, at 3l. 19s. 11d. an ounce.

oz.		£.	s.	d.		oz.	dwt.	
1	:	3	19	11	::	14	8	:
20		20				20		
20		79				288		
		12						
		959						
		288						
		7672						
		7672						
		1918						

2,0)27619,2

13809$\frac{12}{20}$ pence, or

12)13809d. 2$\frac{8}{20}$q.

2,0)115,0s. 9d. 2$\frac{8}{20}$q.

Ans. 57l. 10s. 9d. 2$\frac{8}{20}$q.

EXPLANATION. The three terms being stated by the general rule, as above, the second term is reduced to pence, and the third to penny-weights, these being their lowest denominations, as directed in the first note. The first term is also reduced to dwts. that it may agree with the third, by the same note. The second term is then multiplied by the third, and the product divided by the first, according to the general rule, when the answer comes out 13809 pence, and 12 remaining; which remainder being reduced to farthings, and these divided by the same divisor 20, by the second note, the quotient is 2 farthings, 8 remaining. Lastly, the pence are divided by 12, to reduce them to shillings, and these again by 20 for pounds; when the final sum comes out 57l. 10s. 9d. 2q. for the answer.

2. How much of that in length which is $4\frac{1}{2}$ inches broad, will make a square foot?

Breadth. Length. Breadth.
4·5 : 12 :: 12 :
12

in.
4·5)144·0(32=2f. 8in. the answer.
135

90
90

3. At $10\frac{1}{2}$d. per lb. what is the value of a firkin of butter, containing 56lb.?

lb. d. q. lb.
1 : 10 2 :: 56 :
56=8×7 8

7 0 0
7

£2 9 0 0 the answer.

Or thus:

lb. d. d. lb.
$\frac{1}{1}$: $10\frac{1}{2}=\frac{21}{2}$ :: $\frac{56}{1}$:
$\frac{21}{2}\times\frac{56}{1}=\frac{1176}{2}$=588d.=49s.=2l. 9s. as before.

Or thus:

lb. d. lb.
1 : 10·5 :: 56 :
10·5

280
560

12)588·0
2,0)4,9
£2 9 as before.

4. If $\frac{3}{5}$ of a yard cost $\frac{7}{12}$ of a pound, what will $\frac{6}{15}$ of an English ell cost?

First $\frac{3}{5}$ of a yard $=\frac{3}{5}$ of $\frac{4}{1}$ of $\frac{1}{5}=\frac{3\times4\times1}{5\times1\times5}=\frac{12}{25}$ of an ell.

Then $\frac{12}{25}$ ell : $\frac{7}{12}$l. : : $\frac{6}{15}$ ell :

$$\text{And } \frac{7}{12}\times\frac{6}{15}\times\frac{25}{12}=\frac{7\times\cancel{6}\times\overset{5}{\cancel{25}}}{\underset{2}{\cancel{12}}\times\underset{3}{\cancel{15}}\times12}=\frac{35}{72}$$

$=$9s. 8d. $\frac{2}{3}$ the answer.

5. If $\frac{3}{8}$ of a yard cost $\frac{2}{5}$ of a pound, what will $\frac{1}{4}$ of an English ell cost?

$\frac{3}{8}$ = ·375

$\frac{2}{5}$ = ·4l.

$\frac{1}{4}$ ell$=\frac{5}{16}$yd.= ·3125

·375yd. : ·4l. : : ·3125yd. :

```
        ·3125
        -----
·375)·12500(·333, &c.=6s. 8d. the answer.
      1125
      ----
       1250
       1125
       ----
        1250
        1125
        ----
         125
```

6. What is the value of a cwt. of sugar at 5$\frac{1}{2}$d. per lb.?

Ans. 2l. 11s. 4d.

7. What is the value of a chaldron of coals at 11$\frac{1}{2}$d. per bushel?

Ans. 1l. 14s. 6d.

8. What is the value of a pipe of wine at 10$\frac{1}{2}$d. per pint?

Ans. 44l. 2s.

9. At 3l. 9s. per cwt. what is the value of a pack of wool, weighing 2cwt. 2qrs. 13lb.

Ans. 9l. 6d. $\frac{12}{112}$.

10. What is the value of $1\frac{1}{2}$cwt. of coffee at $5\frac{1}{2}$d. per ounce? Ans. 61l. 12s.

11. Bought 3 casks of raisins, each weighing 2cwt. 2qrs. 25lb. what will they come to at 2l. 1s. 8d. per cwt.?
Ans. 17l. $4\frac{3}{4}$d. $\frac{32}{112}$.

12. What is the value of 2qrs. 1nl. of velvet at 19s. $8\frac{1}{2}$d. per English ell? Ans. 8s. $10\frac{1}{4}$d. $\frac{14}{20}$.

13. Bought 12 pockets of hops, each weighing 1cwt. 2qrs. 17lb.; what do they come to at 4l. 1s. 4d. per cwt.?
Ans. 80l. 12s. $1\frac{1}{2}$d. $\frac{96}{112}$.

14. What is the tax upon 745l. 14s. 8d. at 3s. 6d. in the pound? Ans. 130l. 10s. $0\frac{3}{4}$d. $\frac{48}{240}$.

15. If $\frac{3}{4}$ of a yard of velvet cost 7s. 3d. how many yards can I buy for 13l. 15s. 6d.? Ans. $28\frac{1}{3}$ yards.

16. If an ingot of gold, weighing 9lb. 9oz. 12dwt. be worth 411l. 12s. what is that per grain? Ans. $1\frac{3}{4}$d.

17. How many quarters of corn can I buy for 140 dollars at 4s. per bushel? Ans. 26qrs. 2bu.

18. Bought 4 bales of cloth, each containing 6 pieces, and each piece 27 yards, at 16l. 4s. per piece; what is the value of the whole, and the rate per yard?
Ans. 388l. 16s. at 12s. per yard.

19. If an ounce of silver be worth 5s. 6d. what is the price of a tankard, that weighs 1lb. 10oz. 10dwt. 4gr.?
Ans. 6l. 3s. $9\frac{1}{2}$d. $\frac{96}{480}$.

20. What is the half year's rent of 547 acres of land at 15s. 6d. per acre?
Ans. 211l. 19s. 3d.

21. At 1·75D. per week, how many months' board can I have for 100l.? Ans. 47m. 2w. $\frac{60}{126}$.

22. Bought 1000 Flemish ells of cloth for 90l. how must I sell it per ell in Boston to gain 10l. by the whole?
Ans. 3s. 4d.

23. Suppose a gentleman's income is 1750 dollars a year,

and he spends 19s. 7d. per day, one day with another, how much will he have saved at the year's end?

Ans. 167l. 12s. 1d.

24. What is the value of 172 pigs of lead, each weighing 3cwt. 2qrs. $17\frac{1}{2}$lb. at 8l. 17s. 6d. per fother of $19\frac{1}{2}$cwt.?

Ans. 286l. 4s. $4\frac{1}{2}$d.

25. The rents of a whole parish amount to 1750l. and a rate is granted of 32l. 16s. 6d. what is that in the pound?

Ans. $4\frac{1}{2}$d. $\frac{2880}{420000}$.

26. If keeping for my horse be $11\frac{1}{2}$d. per day, what will be the charge of 11 horses for the year?

Ans. 192l. 7s. $8\frac{1}{2}$d.

27. A person breaking owes in all 1490l. 5s. 10d. and has in money, goods, and recoverable debts, 784l. 17s. 4d. if these things be delivered to his creditors, what will they get in the pound? Ans. 10s. $6\frac{1}{4}$d. $\frac{20993}{35767}$.

28. What must 40s. pay toward a tax, when 652l. 13s. 4d. is assessed at 83l. 12s. 4d.? Ans. 5s. $1\frac{1}{4}$d. $\frac{15376}{15664}$.

29. Bought 3 tuns of oil for 151l. 14s. 85 gallons of which being damaged, I desire to know how I may sell the remainder per gallon, so as neither to gain nor lose by the bargain?

Ans. 4s. $6\frac{1}{4}$d. $\frac{25}{671}$.

30. What quantity of water must I add to a pipe of mountain wine, valued at 33l. to reduce the first cost to 4s. 6d. per gallon? Ans. $20\frac{2}{3}$ gallons.

31. If 15 ells of stuff, $\frac{3}{4}$ yard wide, cost 37s. 6d. what will 40 ells of the same stuff cost, being yard wide?

Ans. 6l. 13s. 4d.

32. Shipped for Barbadoes 500 pairs of stockings at 3s. 6d. per pair, and 1650 yards of baize at 1s. 3d. per yard, and have received in return 348 gallons of rum at 6s. 8d.

per gallon, and 750lb. of indigo at 1s. 4d. per lb. what remains due upon my adventure? Ans. 24l. 12s. 6d.

33. If 100 workmen can finish a piece of work in 12 days, how many are sufficient to do the same in 3 days?

Ans. 400 men.

34. How many yards of matting, 2ft. 6in. broad, will cover a floor, that is 27ft. long, and 20ft. broad?

Ans. 72 yards.

35. How many yards of cloth, 3qrs. wide, are equal in measure to 30 yards, 5qrs. wide? Ans. 50 yards.

36. A borrowed of his friend B 250l. for 7 months, promising to do him the like kindness; sometime after B had occasion for 300l. how long may he keep it to receive full amends for the favor?

Ans. 5 months and 25 days.

37. If, when the price of a bushel of wheat is 6s. 3d. the penny loaf weigh 9oz. what ought it to weigh when wheat is at 8s. $2\frac{1}{4}$d. per bushel? Ans. 6oz. 13dr.

38. If $4\frac{3}{4}$cwt. may be carried 36 miles for 35 shillings, how many pounds can I have carried 20 miles for the same money? Ans. 907lb. $\frac{4}{10}$.

39. How many yards of canvass, that is ell wide, will line 20 yards of say, that is 3qrs. wide? Ans. 12yds.

40. If 30 men can perform a piece of work in 11 days, how many men will accomplish another piece of work, 4 times as big, in a fifth part of the time? Ans. 600.

41. A wall, that is to be built to the height of 27 feet, was raised 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days at the same rate of working? Ans. 36.

42. If $\frac{5}{7}$oz. cost $\frac{11}{12}$l. what will 1oz. cost?

Ans. 1l. 5s. 8d.

43. If $\frac{3}{16}$ of a ship cost 273l. 2s. 6d. what is $\frac{5}{32}$ of her worth? Ans. 227l. 12s. 1d.

44. At $1\frac{1}{2}$l. per cwt. what does $3\frac{1}{3}$lb. come to?

Ans. $10\frac{5}{7}$d.

45. If $\frac{5}{8}$ of a gallon cost $\frac{5}{8}$l. what will $\frac{5}{9}$ of a tun cost?

Ans. 140l.

46. A person, having $\frac{3}{5}$ of a coal mine, sells $\frac{3}{4}$ of his share for 171l. what is the whole mine worth?

Ans. 380l.

47. If, when the days are $13\frac{5}{8}$ hours long, a traveller perform his journey in $35\frac{1}{2}$ days; in how many days will he perform the same journey, when the days are $11\frac{9}{10}$ hours long? Ans. $40\frac{615}{952}$ days.

48. A regiment of soldiers, consisting of 976 men, are to be new clothed, each coat to contain $2\frac{1}{2}$ yards of cloth, that is $1\frac{5}{8}$yd. wide, and to be lined with shalloon, $\frac{7}{8}$yd. wide; how many yards of shalloon will line them?

Ans. 4531yds. 1qr. $2\frac{6}{7}$nl.

PRACTICE.

PRACTICE is a contraction of the Rule of Three, when the first term happens to be an unit, or one; and has its name from its daily use among merchants and tradesmen, being an easy and concise method of working most questions, that occur in trade and business.

The method of proof is by the Rule of Three.

An *aliquot* part of any number is such a part of it, as, being taken a certain number of times, exactly makes that number.

GENERAL RULE.*

1. Suppose the price of the given quantity to be 1l. 1s. or 1d. as is most convenient; then will the quantity itself be the answer, at the supposed price.

2. Divide the given price into aliquot parts, either of the supposed price, or of one another, and the sum of the quotients, belonging to each, will be the true answer required.

NOTE 1. When there is any fractional part, or inferior denomination of the quantity, take the same part of the price, that the given fraction, or inferior denomination, is of the unit, of which the price is given, and add it to the price of the whole number.

NOTE 2. The rule of Practice is nearly superseded by the use of Federal Money.

EXAMPLE.

What is the value of 526 yards of cloth at 3s. 10¼d. per yard?

	526l.			Ans. at 1l.			
3s. 4d. is $\frac{1}{6}$ =	87	13	4	do. at	0	3s.	4d.
4d. is $\frac{1}{10}$ =	8	15	4	do. at			4
2d. is $\frac{1}{2}$ =	4	7	8	do. at			2
¼d. is $\frac{1}{8}$ =	0	10	11½	do. at			0¼
	101	7	3½	do. at		3	10¼ the full price.

Ans. 101l. 7s. 3½d.

---

* The rule will be rendered very evident by an explanation of the example. In this example it is plain, that the quantity 526 is the answer at 1l. consequently, as 3s. 4d. is the $\frac{1}{6}$ of 1l. $\frac{1}{6}$ of that quantity, or 87l. 13s. 4d. is the price at 3s. 4d. In like

By Federal Money.

At $ 0·6423 per yard.

	526
	38538
	12846
	32115
	$ 337·8498 Answer.

2. 8cwt. 2qrs. 16lb. at 2l. 5s. 6d.

	l.	s.	d.
	2	5	6
			8
	18	4	
2qrs. is $\frac{1}{2}$	1	2	9
14lb. is $\frac{1}{4}$		5	$8\frac{1}{4}$
2lb. is $\frac{1}{7}$			$9\frac{3}{4}$
	19l.	13s.	3d. the answer.

3. 5275 yards at 2d. Ans. 43l. 19s. 2d.

4. 1776 yards at 3d. Ans. 22l. 4s.

5. $273\frac{1}{4}$ at 2s. 6d. Ans. 34l. 3s. $1\frac{1}{2}$d.

6. $937\frac{1}{2}$ at 3l. 17s. 8d. Ans. 3640l. 12s. 6d.

---

manner, as 4d. is $\frac{1}{10}$ of 3s. 4d. so $\frac{1}{10}$ of 87l. 13s. 4d. or 8l. 15s. 4d. is the answer at 4d. And by reasoning in this way 4l. 7s. 8d. will be shown to be the price at 2d. and 10s. $11\frac{1}{2}$d. the price at $\frac{1}{4}$.—Now as the sum of all these parts is equal to the whole price (3s. $10\frac{1}{4}$d.) so the sum of the answers, belonging to each price, will be the answer at the full price required. And the same will be true in any example whatever.

TARE AND TRETT.

TARE AND TRETT are practical rules for deducting certain allowances, which are made by merchants and tradesmen in selling their goods by weight.

Tare is an allowance, made to the buyer, for the weight of the box, barrel, or bag, &c. which contains the goods bought, and is either at so much per box, &c. at so much per cwt. or at so much in the gross weight.

Trett is an allowance of 4lb. in every 104lb. for waste, dust, &c.

Cloff is an allowance of 2lb. upon every 3cwt.

Gross weight is the whole weight of any sort of goods, together with the box, barrel, or bag, &c. that contains them.

Suttle is the weight, when part of the allowance is deducted from the gross.

Net weight is what remains after all allowances are made.

CASE 1.

When the tare is a certain weight per box, barrel, or bag, &c.

RULE.*

Multiply the number of boxes, or barrels, &c. by the tare, and subtract the product from the gross, and the remainder is the net weight required.

---

* It is manifest, that this, as well as every other case in this rule, is only an application of the rules of Proportion and Practice.

EXAMPLES.

1. In 7 frails of raisins, each weighing 5cwt. 2qrs. 5lb. gross, tare 23lb. per frail, how much net?

23×7=1cwt. 1qr. 21lb.

cwt.	qrs.	lb.	
5	2	5	
		7	
38	3	7	gross.
1	1	21	tare.
37	1	14	the answer.

2. In 241 barrels of figs, each 3qrs. 19lb. gross, tare 10lb. per barrel, how many pounds net? Ans. 22413.

3. What is the net weight of 14 hogsheads of tobacco, each 5cwt. 2qrs. 17lb. gross, tare 100lb. per hhd.?

Ans. 66cwt. 2qrs. 14lb.

CASE 2.

When the tare is a certain weight per cwt.

RULE.

Divide the gross weight by the aliquot parts of a cwt. contained in the tare, and subtract the quotient from the gross, and the remainder is the net weight.

EXAMPLES.

1. Gross 173cwt. 3qrs. 17lb. tare 16lb. per cwt. how much net?

	cwt.	qrs.	lb.	
	173	3	17	gross.
14lb. is $\frac{1}{8}$	21	2	26	
2lb. is $\frac{1}{7}$	3	0	11	
	24	3	9	
	149	0	8	the answer.

2. What is the net weight of 7 barrels of pot-ash, each weighing 201lb. gross, tare being at 10lb. per cwt.?

Ans. 1281lb. 6oz.

3. In 25 barrels of figs, each 2cwt. 1qr. gross, tare 16lb. per cwt. how much net? Ans. 48cwt. 24lb.

CASE 3.

When Trett is allowed with Tare.

RULE.

Divide the suttle weight by 26, and the quotient is the trett, which subtract from the suttle, and the remainder is the net weight.

EXAMPLES.

1. In 9cwt. 2qrs. 17lb. gross, tare 37lb. and trett as usual, how much net?

cwt.	qrs.	lb.	
9	2	17	gross.
0	1	9	tare.
26)9	1	8	suttle.
	1	11	trett.
8	3	25	the answer.

2. In 7 casks of prunes, each weighing 3cwt. 1qr. 5lb. gross, tare 17½lb. per cwt. and trett as usual, how much net? Ans. 18cwt. 2qrs. 25lb.

3. What is the net weight of 3 hogsheads of sugar weighing as follows: the first, 4cwt. 5lb. gross, tare 73lb. the second, 3cwt. 2qrs. gross, tare 56lb. and the third, 2cwt. 3qrs. 17lb. gross, tare 47lb. and allowing trett to each as usual? Ans. 8cwt. 2qrs. 4lb.

CASE 4.

When tare, trett, and cloff are all allowed.

RULE.

Deduct the tare and trett, as before, and divide the suttle by 168, and the quotient is the cloff, which subtract from the suttle, and the remainder is the net.

EXAMPLES.

1. What is the net weight of a hhd. of tobacco, weighing 15cwt. 3qrs. 20lb. gross, tare 7lb. per cwt. and trett and cloff as usual?

	cwt.	qrs.	lb.	
	15	3	20	gross.
7lb. is $\frac{1}{16}$		3	27	tare.
26)	14	3	21	
		2	8	trett.
168)	14	1	13	suttle.
			9	cloff.
	14	1	4	the answer.

NOTE 2. The first and third terms of each line, if of different denominations, must be reduced to the same denomination.

EXAMPLES.

1. How many men can complete a trench of 135 yards long in 8 days, provided 16 men can dig 54 yards in 6 days?

GENERAL STATING.

$$\left.\begin{array}{l}54 \text{ yds. or } 2\\ 8 \text{ days, or } 4\end{array}\right\} : 16 \text{ men} :: \left\{\begin{array}{l}135 \text{ yds. or } 5\\ 6 \text{ days, or } 3\end{array}\right\} :$$

FIRST METHOD.

	yds. men. yds.	days. men. days.
54÷27=2	2 : 16 : : 5 :	4 : 40 : : 3 :
135÷27=5	5	3
8÷ 2=4	——	——
6÷ 2=3	2)80(40 men.	4)120(30 men, answer.
	8	12
	——	——
	0	0

SECOND METHOD.

$$\left.\begin{array}{c}2\\4\end{array}\right\} : 16 :: \left\{\begin{array}{c}5\\3\end{array}\right\} :$$

```
 8  :  16  : :  15  :
       15
      ----
       80
      16
     -----
   8)240(30 men, the answer as before.
      24
     ----
       0
```

2. If 100l. in one year gain 5l. interest, what will be the interest of 750l. for seven years ?

Ans. 262l. 10s.

3. What principal will gain 262l. 10s. in 7 years, at 5l. per cent. per annum ? Ans. 750l.

4. If a footman travel 130 miles in 3 days, when the days are 12 hours long; in how many days, of 10 hours each, may he travel 360 miles ? Ans. $9\frac{63}{65}$. days.

5. If 120 bushels of corn can serve 14 horses 56 days; how many days will 94 bushels serve 6 horses ?

Ans. $102\frac{16}{45}$ days.

6. If 7oz. 5dwts. of bread be bought at $4\frac{3}{4}$d. when corn is at 4s. 2d. per bushel, what weight of it may be bought for 1s. 2d. when the price of the bushel is 5s. 6d. ?

Ans. 1lb. 4oz. $3\frac{479}{627}$dwts.

7. If the carriage of 13cwt. 1qr. for 72 miles be 2l. 10s. 6d. what will be the carriage of 7cwt. 3qrs. for 112 miles ?

Ans. 2l. 5s. 11d. $1\frac{77}{159}$q.

8. A wall, to be built to the height of 27 feet, was raised to the height of 9 feet by 12 men in 6 days; how many men must be employed to finish the wall in 4 days, at the same rate of working? Ans. 36 men.

9. If a regiment of soldiers, consisting of 939 men, can eat up 351 quarters of wheat in 7 months; how many soldiers will eat up 1464 quarters in 5 months, at that rate ?

Ans. $5483\frac{23}{195}$.

10. If 248 men, in 5 days of 11 hours each, dig a trench 230 yards long, 3 wide and 2 deep; in how many days of 9 hours long, will 24 men dig a trench of 420 yards long, 5 wide and 3 deep ? Ans. $288\frac{59}{207}$.

CONJOINED PROPORTION.

CONJOINED PROPORTION is when the coins, weights, or measures, of several countries are compared in the same question; or it is the joining together of several ratios, and the inferring of the ratio of the first antecedent and the last consequent from the ratios of the several antecedents and their respective consequents.

NOTE 1. The solution of questions, under this rule, may frequently be much shortened by cancelling equal numbers, when in both the columns, or in the first column and third term, and abbreviating those, that are commensurable.

NOTE 2. The proof is by so many statements in the Single Rule of Three, as the nature of the question requires.

CASE 1.

When it is required to find how many of the last kind of coin, weight, or measure, mentioned in the question, are equal to a given number of the first.

RULE.

1. Multiply continually together the antecedents for the first term, and the consequents for the second, and make the given number the third.

2. Then find the fourth term, or proportional, which will be the answer required.

EXAMPLES.

1. If 10lb. at Boston make 9lb. at Amsterdam; 90lb. at Amsterdam, 112lb. at Thoulouse; how many pounds at Thoulouse are equal to 50lb. at Boston?

Ant.		Cons.		
10	:	9		
90	:	112		
900	:	1008	::	50
		50		

)50400(56 the answer.
4500

5400
5400

Or by abbreviation.

10 : 9 :: 50	10 : 1 :: 50	1 : 1 :: 5	
90 : 112	10 : 112*	10 : 112	2 : 112 :: 1 : 56.

56 the answer.

2. If 20 braces at Leghorn be equal to 10 vares at Lisbon; 40 vares at Lisbon to 80 braces at Lucca; how many braces at Lucca are equal to 100 braces at Leghorn?

Ans. 100 braces.

CASE 2.

When it is required to find how many of the first kind of coin, weight, or measure, mentioned in the question, are equal to a given number of the last.

---

* In performing this example, the first abbreviation is obtained by dividing 90 and 9 by their common measure 9; the second by dividing 10 and 50 by their common measure 10; the third by dividing 10 and 5 by their common measure 5; and the fourth, or answer, by dividing 2 and 112 by their common measure 2.

RULE.

Proceed as in the first case, only make the product of the consequents the first term, and that of the antecedents the second.

EXAMPLES.

1. If 100lb. in America make 95lb. Flemish; and 19lb. Flemish, 25lb. at Bolognia; how many pounds in America are equal to 50lb. at Bolognia?

```
Cons.   Ant.
 95  :  100
 25  :   19
----   ----
 475
190
----
2375 :  1900 :: 50 :
          50
        ------
       )95000(40lb. the answer.
        9500
        ------
            0
```

Or by abbreviation.

```
95 : 100          5 : 100        5 : 4
25 :  19 :: 50   25 :   1 :: 50  1 : 1 :: 50   1 : 4 :: 10 :
                                                         4
                                                       ----
                                                  Ans. 40lb.
```

2. If 25lb. at Boston be 22lb. at Nuremburg; 88lb. at Nuremburg, 92lb. at Hamburgh; 46lb. at Hamburgh, 49lb. at Lyons; how many pounds at Boston are equal to 98lb. at Lyons?

Ans. 100lb.

3. If 6 braces at Leghorn make 3 ells English; 5 ells English, 9 braces at Venice; how many braces at Leghorn will make 45 braces at Venice? Ans. 50 braces.*

FELLOWSHIP.

Fellowship is a general rule, by which merchants, &c. trading in company, with a joint stock, determine each person's particular share of the gain or loss in proportion to his share in the joint stock.

By this rule a bankrupt's estate may be divided among his creditors; as also legacies adjusted, when there is a deficiency of assets or effects.

SINGLE FELLOWSHIP.

Single Fellowship is when different stocks are employed for any certain equal time.

RULE.*

As the whole stock is to the whole gain or loss, so is each

---

* *Barter* is the exchanging of one commodity for another, and directs traders so to proportion their goods, that neither party may sustain loss.

Loss and Gain is a rule, that discovers what is got or lost in the buying or selling of goods; and instructs merchants and traders to raise or lower the price of their goods, so as to gain or lose a certain sum per cent. &c.

Questions in these rules are performed by the Rule of Three.

* That the gain or loss, in this rule, is in proportion to their

man's particular stock to his particular share of the gain or loss.

METHOD OF PROOF.

Add all the shares together, and the sum will be equal to the gain or loss, when the question is right.

EXAMPLES.

1. Two persons trade together; A put into stock $130 and B $220, and they gained $500; what is each person's share thereof?*

```
130
220
———
350 : 500 :: 130
               500
         ————————
   35,0)6500,0(185·71 15/35
        35
        ——
        300
        280
        ———
         200
         175
         ———
          250
          245
          ———
           50
           35
           ——
           15
```

---

stocks is evident: for, as the times the stocks are in trade are equal, if I put in ½ of the whole stock, I ought to have ½ of the

350 : 500 :: 220 :

500

35,0)11000,0(314·28$\frac{20}{35}$

105

50

35

150

140

100

70

300

280

20

\$185·71$\frac{15}{35}$=A's share.

314·28$\frac{20}{35}$=B's share

\$500·00 the proof.

2. A and B have gained by trading \$182. A put into stock \$300 and B \$400 ; what is each person's share of the profit? Ans. A \$78 and B \$104.

---

whole gain ; if my part of the whole stock be $\frac{1}{3}$, my share of the whole gain or loss ought to be $\frac{1}{3}$ also. And generally, if I put in $\frac{1}{n}$ of the stock, I ought to have $\frac{1}{n}$ part of the whole gain or loss ; that is, the same ratio, that the whole stock has to the whole gain or loss, must each person's particular stock have to his particular gain or loss.

3. Divide $120 between three persons, so that their shares shall be to each other as 1, 2, and 3 respectively.

Ans. $20, $40, and $60.

4. Three persons make a joint stock. A put in $185·66, B $98·50, and C $76·85; they trade and gain $222; what is each person's share of the gain?

Ans. A $104·17$\frac{83}{36101}$, B $60·57$\frac{6243}{36101}$, & C $47·25$\frac{29775}{36101}$.

5. Three merchants, A, B, and C, freight a ship with 340 tuns of wine; A loaded 110 tuns, B 97, and C the rest. In a storm the seamen were obliged to throw 85 tuns overboard; how much must each sustain of the loss?

Ans. A 27$\frac{1}{2}$, B 24$\frac{1}{4}$, and C 33$\frac{1}{4}$.

6. A ship worth $860 being entirely lost, of which $\frac{1}{8}$ belonged to A, $\frac{1}{4}$ to B, and the rest to C; what loss will each sustain, supposing $500 of her to be insured?

Ans. A $45, B $90, and C $225.

7. A bankrupt is indebted to A $277·33, to B $305·17, to C $152, and to D $105. His estate is worth only $677·50; how must it be divided?

Ans. A $223·81$\frac{2580}{8395}$, B $246·28$\frac{615}{8395}$,
C $122·66$\frac{6930}{8395}$, and D $84·73$\frac{6665}{8395}$.

8. A and B, venturing equal sums of money, clear by joint trade $154. By agreement A was to have 8 per cent. because he spent his time in the execution of the project, and B was to have only 5 per cent.; what was A allowed for his trouble? Ans. $35·53$\frac{11}{13}$.

DOUBLE FELLOWSHIP.

Double Fellowship is when different or equal stocks are employed for different times.

RULE.*

Multiply each man's stock into the time of its continuance, then say,

As the total sum of all the products is to the whole gain or loss,

So is each man's particular product to his particular share of the gain or loss.

EXAMPLES.

1. A and B hold a piece of ground in common, for which they are to pay $36. A put in 23 oxen for 27 days, and B 21 oxen for 35 days; what part of the rent ought each man to pay?

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* Mr. MALCOM, Mr. WARD, and several other authors have given an analytical investigation of this rule; but the most general and elegant method perhaps is that, which Dr. HUTTON has given in his Arithmetic, namely,

When the times are equal, the shares of the gain or loss are evidently as the stocks, as in Single Fellowship; and when the stocks are equal, the shares are as the times; wherefore, when neither are equal, the shares must be as their products.

```
23×27=621     1356 : 36 : : 621 :
21×35=735            621
     ----            ---
     1356             36
                     72
                    216
                    -----
                1356)22356(16·48 912/1356
                    1356
                    ----
                     8796
                     8136
                     ----
                      6600
                      5424
                      -----
                      11760
                      10848
                      -----
                        912

1356 : 36 : : 735 :
       735
       ---
       180
      108
     252
     ----
1356)26460(19·51 444/1356
     1356
     ----
     12900
     12204
     -----
       6960        $16.48 912/1356=A's share.
       6780         19·51 444/1356=B's share.
       ----         -----
        1800       $36·00 the proof.
        1356        -----
        ----
         444
```

2. Three graziers hired a piece of land for \$60·50. A put in 5 sheep for $4\frac{1}{2}$ months, B put in 8 for 5 months, and C put in 9 for $6\frac{1}{2}$ months; how much must each pay of the rent? Ans. A \$11·25, B \$20, and C \$29·25.

3. Two merchants enter into partnership for 18 months; A put into stock at first \$200, and at the end of 8 months he put in \$100 more; B put in at first \$550, and at the end of 4 months took out \$140. Now at the expiration of the time they find they have gained \$526; what is each man's just share? Ans. A's \$192·95$\frac{70}{1254}$.
B's 333·04$\frac{1184}{1254}$.

4. A, with a capital of \$1000 began trade January 1, 1776, and meeting with success in business he took in B as a partner, with a capital of \$1500 on the first of March following. Three months after that they admit C as a third partner, who brought into stock \$2800, and after trading together till the first of the next year, they find the gain, since A commenced business, to be \$1776·50. How must this be divided among the partners?

Ans. A's \$457·46$\frac{364}{466}$.
B's 571·83$\frac{222}{466}$.
C's 747·19$\frac{346}{466}$.

ALLIGATION.

Alligation teaches how to mix several simples of different qualities, so that the composition may be of a middle quality; and is commonly distinguished into two principal cases, called *Alligation medial* and *Alligation alternate*.

ALLIGATION MEDIAL.

Alligation medial is the method of finding the rate of the compound, from having the rates and quantities of the several simples given.

RULE.*

Multiply each quantity by its rate; then divide the sum of the products by the sum of the quantities, or the whole composition, and the quotient will be the rate of the compound required.

EXAMPLES.

1. Suppose 15 bushels of wheat at 5s. per bushel, and 12 bushels of rye at 3s. 6d. per bushel were mixed together; how must the compound be sold per bushel without loss or gain?

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* The truth of this rule is too evident to need a demonstration.

NOTE. If an ounce or any other quantity of pure gold be reduced into 24 equal parts, these parts are called carats; but gold is often mixed with some baser metal, which is called the alloy, and the mixture is said to be of so many carats fine, according to the proportion of pure gold contained in it; thus, if 22 carats of pure gold and 2 of alloy be mixed together, it is said to be 22 carats fine.

If any one of the simples be of little or no value with respect to the rest, its rate is supposed to be nothing, as water mixed with wine, and alloy with gold or silver.

```
60          42          15
15          12          12
---         ---         --
300         504         27
60          900
---         ---
900     27)1404(52d.=4s. 4d. the answer.
           135
           ---
            54
            54
            --
```

2. A composition being made of 5lb. of tea at 7s. per pound, 9lb. at 8s. 6d. per pound, and $14\frac{1}{2}$lb. at 5s. 10d. per pound, what is a pound of it worth? Ans. 6s. $10\frac{1}{2}$d.

3. Mixed 4 gallons of wine at 4s. 10d. per gallon, with 7 gallons at 5s. 3d. per gallon, and $9\frac{3}{4}$ gallons at 5s. 8d. per gallon; what is a gallon of this composition worth?

Ans. 5s. $4\frac{1}{4}$d.

4. A goldsmith melts 8lb. $5\frac{1}{2}$oz. of gold bullion of 14 carats fine, with 12lb. $8\frac{1}{2}$oz. of 18 carats fine; how many carats fine is this mixture? Ans. $16\frac{204}{508}$ carats.

5. A refiner melts 10lb. of gold of 20 carats fine with 16lb. of 18 carats fine; how much alloy must he put to it to make it 22 carats fine?

Ans. It is not fine enough by $3\frac{6}{26}$ carats, so that no alloy must be put to it, but more gold.

ALLIGATION ALTERNATE.

Alligation alternate is the method of finding what quantity of any number of simples, whose rates are given, will compose a mixture of a given rate; so that it is the reverse of Alligation medial, and may be proved by it.

RULE 1.*

1. Write the rates of the simples in a column under each other.

2. Connect or link with a continued line the rate of each simple, which is less than that of the compound, with one or any number of those, that are greater than the compound; and each greater rate with one or any number of the less.

3. Write the difference between the mixture rate and

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* DEMONSTRATION. By connecting the less rate to the greater, and placing the differences between them and the mean rate alternately, the quantities resulting are such, that there is precisely as much gained by one quantity as is lost by the other, and therefore the gain and loss upon the whole are equal, and are exactly the proposed rate; and the same will be true of any other two simples, managed according to the rule.

In like manner, let the number of simples be what it may, and with how many soever each is linked, since it is always a less with a greater than the mean price, there will be an equal balance of loss and gain between every two, and consequently an equal balance on the whole. Q. E. D.

It is obvious from the rule, that questions of this sort admit of a great variety of answers; for having found one answer, we may find as many more as we please, by only multiplying or dividing each of the quantities found by 2, 3, or 4, &c. the reason of which is evident; for, if two quantities of two simples make a balance of loss and gain, with respect to the mean price, so must also the double or treble, the $\frac{1}{2}$ or $\frac{1}{3}$ part, or any other ratio of these quantities, and so on, *ad infinitum*.

Questions of this kind are called by algebraists *indeterminate* or *unlimited* problems, and by an analytical process theorems may be raised, that will give all the *possible* answers.

that of each of the simples opposite to the rates, with which they are respectively linked.

4. Then if only one difference stand against any rate, it will be the quantity belonging to that rate; but if there be several, their sum will be the quantity.

EXAMPLES.

1. A merchant would mix wines at 14s. 19s. 15s. and 22s. per gallon, so that the mixture may be worth 18s. the gallon; what quantity of each must be taken?

	Rates	Quantities
18	14	4 at 14s.
	15	1 at 15s.
	19	3 at 19s.
	22	4 at 22s.

Or thus:

	Rates	Differences	Quantities
18	14	1+4	5 at 14s.
	15	1	1 at 15s.
	19	4+3	7 at 19s.
	22	4	4 at 22s.

2. How much wine at 6s. per gallon and at 4s. per gallon must be mixed together, that the composition may be worth 5s. per gallon?

Ans. 12 gallons, or equal quantities of each.

3. How much corn at 2s. 6d. 3s. 8s. 4s. and 4s. 8d. per bushel must be mixed together, that the compound may be worth 3s. 10d. per bushel?

Ans. 12 at 2s. 6d. 12 at 3s. 8d. 18 at 4s. and 18 at 4s. 8d.

4. A goldsmith has gold of 17, 18, 22, and 24 carats fine; how much must be taken of each to make it 21 carats fine?

Ans. 3 of 17, 1 of 18, 3 of 22, and 4 of 24.

5. It is required to mix brandy at 8s. wine at 7s. cider at 1s. and water at 0 per gallon together, so that the mixture may be worth 5s. per gallon?

Ans. 9 of brandy, 9 of wine, 5 of cider, and 5 of water.

RULE 2.*

When the whole composition is limited to a certain quantity, find an answer as before by linking; then say as the sum of the quantities, or differences thus determined, is to the given quantity, so is each ingredient, found by linking, to the required quantity of each.

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* A great number of questions might be here given relating to the specific gravity of metals, &c. but one of the most curious, with the operation at large, may serve as a sufficient specimen.

HIERO, king of Syracuse, gave orders for a crown to be made entirely of pure gold; but suspecting the workmen had debased it by mixing it with silver or copper, he recommended the discovery of the fraud to the famous ARCHIMEDES; and desired to know the exact quantity of alloy in the crown.

ARCHIMEDES, in order to detect the imposition, procured two other masses, one of pure gold, the other of silver or copper, and each of the same weight with the former; and each being put separately into a vessel full of water, the quantity of water expelled by them determined their specific bulks; from which and their given weights, the exact quantities of gold and alloy in the crown may be determined.

Suppose the weight of each crown to be 10lb. and that the water expelled by the copper or silver was 92lb. by the gold 52lb. and by the compound crown 64lb. what will be the quantities of gold and alloy in the crown?

EXAMPLES.

1. How many gallons of water at 0s. per gallon, must be mixed with wine worth 3s. per gallon, so as to fill a vessel of 100 gallons, and that a gallon may be afforded at 2s. 6d.?

$$30\begin{cases}0 & 6\\ 36 & 30\end{cases}$$

$$\begin{array}{r}6\\30\\\hline 36\end{array}$$

36 : 100 : : 6 :   36 : 100 : : 30 :

$$\begin{array}{r}100\\6\\\hline 36)600(16\\36\\\hline 240\\216\\\hline 24\end{array}\qquad\qquad\begin{array}{r}100\\30\\\hline 36)3000(83\\288\\\hline 120\\108\\\hline 12\end{array}$$

Ans. $83\frac{12}{36}$ gallons of wine, and $16\frac{24}{36}$ of water.

2. A grocer has currants at 4d. 6d. 9d. and 11d. per lb. and he would make a mixture of 240lb. so that it may be afforded at 8d. per pound; how much of each sort must he take?

Ans. 72lb. at 4d. 24 at 6d. 48 at 9d. and 96 at 11d.

The rates of the simples are 92 and 52, and of the compound 64; therefore

$$64\begin{cases}92 & 12 \text{ of copper,}\\ 52 & 28 \text{ of gold.}\end{cases}$$

And the sum of these is 12+28=40, which should have been but 10; whence, by the rule,

40 : 10 : : 12 : 3lb. of copper, } the answer.
40 : 10 : : 28 : 7lb. of gold, }

3. How much gold of 15, of 17, of 18, and of 22 carats fine must be mixed together to form a composition of 40 ounces of 20 carats fine?

Ans. 5oz. of 15, of 17, and of 18, and 25 of 22.

RULE. 3.*

When one of the ingredients is limited to a certain quantity; take the difference between each price and the mean rate as before; then,

As the difference of that simple, whose quantity is given, is to the rest of the differences severally, so is the quantity given to the several quantities required.

EXAMPLES.

1. How much wine at 5s. at 5s. 6d. and 6s. the gallon must be mixed with 3 gallons at 4s. per gallon, so that the mixture may be worth 5s. 4d. per gallon?

64	48	8+2=10
	60	8+2=10
	66	16+4=20
	72	16+4=20

10 : 10 :: 3 : 3
10 : 20 :: 3 : 6
10 : 20 :: 3 : 6

Ans. 3 gallons at 5s. 6 at 5s. 6d. and 6 at 6s.

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* In the very same manner questions may be wrought, when several of the ingredients are limited to certain quantities, by finding first for one limit and then for another.

The two last rules can want no demonstration, as they evidently result from the first, the reason of which has been already explained.

2. A grocer would mix teas at 12s. 10s. and 6s. with 20 lb. at 4s. per pound; how much of each sort must he take to make the composition worth 8s. per. lb.?

Ans. 20lb. at 4s. 10 at 6s. 10 at 10s. and 20 at 12s.

3. How much gold of 15, of 17, and of 22 carats fine, must be mixed with 5oz. of 18 carats fine, so that the composition may be 20 carats fine?

Ans. 5oz. of 15 carats fine, 5 of 17, and 25 of 22.

INVOLUTION.

A Power is a number produced by multiplying any given number continually by itself a certain number of times.

Any number is itself called the *first power*; if it be multiplied by itself, the product is called the *second power*, or the *square*; if this be multiplied by the first power again, the product is called the *third power*, or the *cube*; and if this be multiplied by the first power again, the product is called the *fourth power*, or *biquadrate*; and so on; that is, the power is denominated from the number, which exceeds the multiplication by 1.

Thus, 3 is the first power of 3.
$3\times3=9$ is the second power of 3.
$3\times3\times3=27$ is the third power of 3.
$3\times3\times3\times3=81$ is the fourth power of 3.
&c. &c.

And in this manner is calculated the following table of powers.

TABLE *of the first twelve Powers of the* **9** *Digits.*

1st Pow.	1	2	3	4	5	6	7	8	9
2d Pow.	1	4	9	16	25	36	49	64	81
3d Pow.	1	8	27	64	125	216	343	512	729
4th Pow.	1	16	81	256	625	1266	2401	4096	6561
5th Pow.	1	32	243	1024	3125	7776	16807	32768	59049
6th Pow.	1	64	729	4096	15625	46656	117649	262144	531441
7th Pow.	1	128	2187	16384	78125	279936	823543	2097152	4782969
8th Pow.	1	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th Pow.	1	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th Pow.	1	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
11th Pow.	1	2048	177147	4194304	48828125	362797056	1977326743	8589934592	31381059609
12th Pow.	1	4096	531441	16777216	244140625	2176782336	13841287201	68719476736	282420536481

NOTE 1. The number, which exceeds the multiplications by 1, is called the *index*, or *exponent*, of the power; so the index of the first power is 1, that of the second power is 2, and that of the third is 3, &c.

NOTE 2. Powers are commonly denoted by writing their indices above the first power; so the second power of 3 may be denoted thus 3^2, the third power thus 3^3, the fourth power thus 3^4, &c. and the sixth power of 503 thus 503^6.

Involution is the finding of powers; to do which we have evidently the following

RULE.

Multiply the given number, or first power, continually by itself, till the number of multiplications be 1 less than the index of the power to be found, and the last product will be the power required.*

* NOTE. The raising of powers will be sometimes shortened by working according to this observation, viz. whatever two or more powers are multiplied together, their product is the power, whose index is the sum of the indices of the factors; or if a power be multiplied by itself, the product will be the power, whose index is double of that, which is multiplied: so if I would find the sixth power, I might multiply the given number twice by itself for the third power, then the third power into itself would give the sixth power; or if I would find the seventh power, I might first find the third and fourth, and their product would be the seventh; or lastly, if I would find the eighth power, I might first find the second, then the second into itself would be the fourth, and this into itself would be the eighth.

NOTE. Whence, because fractions are multiplied by taking the products of their numerators and of their denominators, they will be involved by raising each of their terms to the power required. And if a mixed number be proposed, either reduce it to an improper fraction, or reduce the vulgar fraction to a decimal, and proceed by the rule.

EXAMPLES.

1. What is the second power of 45? Ans. 2025.

2. What is the square of ·027? Ans. ·000729.

3. What is the third power of 3·5? Ans. 42·875.

4. What is the fifth power of ·029?
Ans. ·0000000205111149.

5. What is the sixth power of 5·03?
Ans. 16196·005304479729.

6. What is the second power of $\frac{2}{3}$? Ans. $\frac{4}{9}$.

EVOLUTION.

THE ROOT of any given number, or power, is such a number as, being multiplied by itself a certain number of times, will produce the power; and it is denominated the *first*, *second*, *third*, *fourth*, *&c. root* respectively, as the number of multiplications, made of it to produce the given power, is 0, 1, 2, 3, &c. that is, the name of the root is taken from the number, which exceeds the multiplications by 1, like the name of the power in Involution.

NOTE 1. The *index* of the root, like that of the power in Involution, is 1 more than the number of multiplications, necessary to produce the power or given number.

NOTE 2. Roots are sometimes denoted by writing $\surd$ before the power, with the index of the root against it: so the third root of 50 is $\sqrt[3]{50}$, and the second root of it is $\sqrt{50}$, the index 2 being omitted, which index is always understood, when a root is named or written without one. But if the power be expressed by several numbers with the sign + or —, &c. between them, then a line is drawn from the top of the sign of the root, or radical sign, over all the parts of it: so the third root of 47 — 15 is $\sqrt[3]{47 - 15}$. And sometimes roots are designed like powers, with the reciprocal of the index of the root above the given number. So the second root of 3 is $3^{\frac{1}{2}}$; the second root of 50 is $50^{\frac{1}{2}}$; and the third root of it is $50^{\frac{1}{3}}$; also the third root of 47 — 15 is $\overline{47-15}|^{\frac{1}{3}}$. And this method of notation has justly prevailed in the modern algebra; because such roots, being considered as fractional powers, need no other directions for any operations to be made with them, than those for integral powers.

NOTE 3. A number is called a *complete* power of any kind, when its root of the same kind can be accurately extracted; but if not, the number is called an *imperfect* power, and its root a *surd* or *irrational* number: so 4 is a complete power of the second kind, its root being 2; but an imperfect power of the third kind, its root being a surd number.

Evolution is the finding of the roots of numbers either accurately, or in decimals, to any proposed extent.

The power is first to be prepared for extraction, or evolution, by dividing it from the place of units, to the left in integers, and to the right in decimal fractions, into periods, each containing as many places of figures, as are denominated by the index of the root, if the power contain a complete number of such periods: if it do not, the defect will be either on the right, or left, or both; if the defect be on the right, it may be supplied by annexing cyphers, and after this, whole periods of cyphers may be annexed to continue the extraction, if necessary; but if there be a defect on the left, such defective period must remain unaltered, and is accounted the first period of the given number, just the same, as if it were complete.

Now this division may be conveniently made by writing a point over the place of units, and also over the last figure of every period on both sides of it; that is, over every second figure, if it be the second root; over every third, if it be the third root, &c.

Thus, to point this number	21035896·12735;
for the second root, it will be	21035896·127350;
but for the third root	21035896·127350;
and for the fourth	21035896·12735000.

NOTE. The root will contain just as many places of figures, as there are periods or points in the given power; and they will be integers or decimals respectively, as the periods are so, from which they are found, or to which they correspond; that is, there will be as many integral or decimal figures in the root, as there are periods of integers or decimals in the given number.

TO EXTRACT THE SQUARE ROOT.

RULE.*

1. Having distinguished the given number into periods, find a square number by the table or trial, either equal to, or next less than the first period, and put the root of it on the right of the given number, in the manner of a quotient figure in Division, and it will be the first figure of the root required.

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* In order to show the reason of the rule, it will be proper to premise the following

LEMMA. The product of any two numbers can have at most but as many places of figures, as are in both the factors, and at least but one less.

DEMONSTRATION. Take two numbers, consisting of any number of places, but let them be the least possible of those places, namely, unity with cyphers, as 1000 and 100; then their product will be 1 with as many cyphers annexed, as are in both the numbers, namely, 100000; but 100000 has one place less than 1000 and 100 together have; and since 1000 and 100 were taken the least possible, the product of any other two numbers, of the same number of places, will be greater than 100000; consequently the product of any two numbers can have at least but one place less than both the factors.

Again, take two numbers of any number of places, that shall be the greatest of these places possible, as 999 and 99. Now 999×99 is less than 999×100; but 999×100 (=99900) contains only as many places of figures, as are in 999 and 99; therefore 999×99, or the product of any other two numbers, consisting of

2. Subtract the assumed square from the first period, and to the remainder bring down the next period for a dividend.

3. Place the double of the root, already found, on the left of the dividend for a divisor.

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the same number of places, cannot have more places of figures than are in both its factors.

COROLLARY 1. A square number cannot have more places of figures than double the places of the root, and at least but one less.

COR. 2. A cube number cannot have more places of figures than triple the places of the root, and at least but two less.

The truth of the rule may be shown algebraically thus:

Let N= the number, whose square root is to be found.

Now it appears from the lemma, that there will be always as many places of figures in the root, as there are points or periods in the given number, and therefore the figures of those places may be represented by letters.

Suppose N to consist of two periods, and let the figures in the root be represented by a and b.

Then $\overline{a+b}^2 = a^2+2ab+b^2 = N =$ given number; and to find the root of N is the same, as finding the root of $a^2+2ab+b^2$, the method of doing which is as follows:

1st divisor $a)a^2+2ab+b^2(a+b=$ root.

a^2

2d divisor $2a+b)2ab+b^2$

$2ab+b^2$

Again suppose N to consist of 3 periods, and let the figures of the root be represented by a, b, and c.

4. Consider what figure must be annexed to the divisor, so that if the result be multiplied by it, the product may be equal to, or next less than the dividend, and it will be the second figure of the root.

5. Subtract the said product from the dividend, and to the remainder bring down the next period for a new dividend.

6. Find a divisor as before, by doubling the figures already in the root; and from these find the next figure of the root, as in the last article; and so on through all the periods to the last.

NOTE 1. When the root is to be extracted to a great number of places, the work may be much abbreviated thus: having proceeded in the extraction by the common method till you have found one more than half the required number of figures in the root, the rest may be found by dividing the last remainder by its corresponding divisor, annexing a cypher to every dividual, as in division of decimals; or rather,

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Then $\overline{a+b+c}^2 = a^2+2ab+b^2+2ac+2bc+c^2$, and the manner of finding a, b, and c will be, as before: thus,

1st divisor $a)a^2+2ab+b^2+2ac+2bc+c^2(a+b+c=$ root.
a^2

2d divisor $\overline{2a+b})2ab+b^2$
$2ab+b^2$

3d divisor $2a+2b+c)2ac+2bc+c^2$
$2ac+2bc+c^2$

Now the operation in each of these cases exactly agrees with the rule, and the same will be found to be true, when N consists of any number of periods whatever.

without annexing cyphers, by omitting continually the first figure of the divisor on the right, after the manner of contraction in division of decimals.

Note 2. By means of the square root we readily find the fourth root, or the eighth root, or the sixteenth root, &c. that is, the root of any power, whose index is some power of the number 2; namely, by extracting so often the square root, as is denoted by that power of 2; that is, twice for the fourth root, thrice for the eighth root, and so on.

TO EXTRACT THE SQUARE ROOT OF A VULGAR FRACTION.

RULE.

First prepare all vulgar fractions by reducing them to their least terms, both for this and all other roots. Then

1. Take the root of the numerator and that of the denominator for the respective terms of the root required. And this is the best way, if the denominator be a complete power. But if not, then

2. Multiply the numerator and denominator together; take the root of the product: this root, being made the numerator to the denominator of the given fraction, or the denominator to the numerator of it, will form the fractional root required.

$$\text{That is, } \sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b} = \frac{a}{\sqrt{ab}}.$$

And this rule will serve, whether the root be finite or infinite.

Or 3. Reduce the vulgar fraction to a decimal, and extract its root.

EXAMPLES.

1. Required the square root of 5499025.

```
 .  . . .
 5499025(2345 the root.
 4
 ----
43|149
 3|129
 ---------
464|2090
  4|1856
 ----------
4685|23425
    |23425
 ----------
```

2. Required the square root of 184·2.

```
 .  .  . .
 184·2000(13·57 the root.
 1
 ----
23|84
 3|69
 ---------
265|1520
  5|1325
 ----------
2707|19500
    |18949
     ------
       551 remainder.
```

3. Required the square root of 2 to 12 places.

```
       2(1·41421356237 + root.
        2
       ——
       24|100
        4| 96
       ———————
      281|400
        1|281
       ————————
     2824|11900
        4|11296
       —————————
    28282|30400
        2|56564
       ——————————
   282841|383600
        1|282841
       ———————————
  2828423|10075900
        3| 8485269
       ———————————
  2828426)1590631(56237 +
   ·····  1414213
          ———————
           176418
           169706
           ———————
             6712
             5657
             ————
             1055
              849
             ————
              206
              198
             ————
                8
```

4. What is the square root of 152399025

Ans. 12345.

5. What is the square root of ·00032754?

Ans. ·01809.

6. What is the square root of $\frac{5}{12}$? Ans. ·645497.

7. What is the square root of $6\frac{2}{5}$? Ans. 2·5298, &c.

8. What is the square root of 10? Ans. 3·162277, &c.

TO EXTRACT THE CUBE ROOT.

RULE.*

1. Having divided the given number into periods of 3 figures, find the nearest less cube to the first period by the table of powers or trial; set its root in the quotient, and subtract the said cube from the first period; to the remainder bring down the second period, and call this the *resolvend.*

2. To three times the square of the root, just found, add three times the root itself, setting this one place more to the right than the former, and call this sum the *divisor.* Then divide the resolvend, wanting the last figure, by the divisor,

---

* The reason of pointing the given number, as directed in the rule, is obvious from Cor. 2, to the Lemma, used in demonstrating the Square Root; and the rest of the operation will be best understood from the following analytical process.

Suppose N, the given number, to consist of two periods, and let the figures in the root be denoted by a and b.

Then $\overline{a+b}|^3 = a^3+3a^2b+3ab^2+b^3 = N =$ given number, and to find the cube root of N is the same as to find the cube root of $a^3+3a^2b+3ab^2+b^2$; the method of doing which is as follows:

for the next figure of the root, which annex to the former; calling this last figure *e*, and the part of the root before found call *a*.

3. Add together these three products, namely, thrice the square of *a* multiplied by *e*, thrice *a* multiplied by the square of *e*, and the cube of *e*, setting each of them one place farther toward the right than the former, and call the sum the *subtrahend;* which must not exceed the resolvend; and if it do, then make the last figure *e* less, and repeat the operation for finding the subtrahend.

4. From the resolvend take the subtrahend, and to the remainder join the next period of the given number for a new resolvend; to which form a new divisor from the whole root now found; and thence another figure of the root, as before, &c.

$a^3+3a^2b+3ab^2+b^3$ $(a+b=$ root.

a^3

$3a^2b+3ab^2+b^3$ resolvend.

$3a^2$

$\quad +3a$

$3a^2 \; +3a$ divisor.

$3a^2b$

$\quad +3ab^2$

$\qquad +b^3$

$3a^2b+3ab^2+b^3$

* * *

And in the same manner may the root of a quantity, consisting of any number of periods whatever, be found.

EXAMPLES.

1. To extract the cube root of 48228·544.

$3 \times 3^2 = 27$ | 48228·544(36·4 root.
$3 \times 3 = 09$ | 27

Divisor 279 | 21228 resolvend.

$3 \times 3^2 \times 6 = 162$
$3 \times 3 \times 6^2 = 324$ } add
$6^3 = 216$

$3 \times 36^2 = 3888$ | 19656 subtrahend.
$3 \times 36 = 108$ |

38988 | 1572544 resolvend.

$3 \times 36^2 \times 4 = 15552$
$3 \times 36 \times 4^2 = 1728$ } add
$4^3 = 64$

1572544 subtrahend.

2. What is the cube root of 1092727? Ans. 103.

3. What is the cube root of 27054036008? Ans. 3002.

4. What is the cube root of ·0001357?

Ans. ·05138, &c.

5. What is the cube root of $\frac{1520}{5130}$? Ans. $\frac{2}{3}$.

6. What is the cube root of $\frac{2}{3}$? Ans. ·873 &c.

RULE FOR EXTRACTING THE CUBE ROOT BY APPROXIMATION.*

1. Find by trial a cube near to the given number, and call it the *supposed cube.*

2. Then twice the supposed cube added to the given number is to twice the given number added to the supposed cube, as the root of the supposed cube is to the root required nearly. Or as the first sum is to the difference of the given and supposed cube, so is the supposed root to the difference of the roots nearly.

3. By taking the cube of the root thus found for the supposed cube, and repeating the operation, the root will be had to a still greater degree of exactness.

---

* That this rule converges extremely fast may be easily shown thus:

Let N= given number, a^3= supposed cube, and x= correction.

Then $2a^3+N : 2N+a^3 :: a : a+x$ by the rule, and consequently $\overline{2a^3+N}\times\overline{a+x}=\overline{2N+a^3}\times a$, or $\overline{2a^3+\overline{a+x}^3}\times\overline{a+x}=\overline{2N+a^3}\times a$.

Or $2a^4+2a^3x+a^4+4a^3x+6a^2x^2+4ax^3+x^4=2aN+a^4$, and by transposing the terms, and dividing by $2a$.

$N=a^3+3a^2x+3ax^2+x^3+x^3+\frac{x^4}{2a}$, which by neglecting the terms $x^3+\frac{x^4}{2a}$, as being very small, becomes $N=a^3+3a^2x+3ax^2+x^3$= the known cube of $a+x$. Q. E. I.

EXAMPLES.

1. It is required to find the cube root of 98003449.

Let 125000000 = supposed cube, whose root is 500;

Then 125000000		98003449			
2		2			
250000000		196006898			
98003449		125000000			
348003449	:	321006898	: :	500	:
		500			

[root nearly.

348003449)160503449000(461=corrected root, or
1392013796
2130206940
2088020694
421862460
348003449
73859011

2. Required the cube root of 21035·8.

Here we soon find that the root lies between 20 and 30, and then between 27 and 28. Therefore 27 being taken, its cube is 19683 the assumed cube. Then

19683	21035·8
2	2
39366	42071·6
21035·8	19683

As 60401·8 : 61754·6 : : 27 : 27·6047

```
              61754·6
                   27
              -------
              4322822
             1235092
             --------
             1667374·2
60401·8)1667374·2(27·6047 the root nearly.
        1208036
        -------
         459338
         422813
         ------
          36525
          36241
          -----
            284
            242
            ---
             42
```

Again for a second operation, the cube of this root is 21035·318645155823, and the process by the latter method is thus:

```
21035·318645, &c.
               2
------------------
42070·637290       21035·8
210358             21035·318645, &c.
-----------        -----------------
```

As 63106·43729 : diff. ·481355 : : 27·6047 : the diff.= ·000210834

```
 ·000210834
-------------
27·604910834=
```

the root required.

3. What is the cube root of 157464? Ans. 54.

4. What is the cube root of $\frac{4}{9}$? Ans. ·763, &c.

5. What is the cube root of 117? Ans. 4·89097.

TO EXTRACT THE ROOTS OF POWERS IN GENERAL.

RULE.*

1. Prepare the given number for extraction by pointing off from the units place as the root required directs.

2. Find the first figure of the root by trial, and subtract its power from the given number.

---

* This rule will be sufficiently obvious from the work in the following example.

Extract the cube root of $a^6+6a^5-40a^3+96a-64$.

$$a^6+6a^5-40a^3+96a-64(a^2+2a-4$$

$$a^6$$

$$3a^4)6a^5(+2a$$

$$a^6+6a^5+12a^4+8a^3=\overline{a^2+2a}^3$$

$$\overline{a^2+2a}^2\times3=3a^4+12a^3+12a^2)-12a^4-48a^3+96a-64(-4$$

$$a^6+6a^5-40a^3+96a-64=\overline{a^2+2a-4}^3$$

* *

When the index of the power, whose root is to be extracted, is a composite number, the following rule will be serviceable:

Take any two or more indices, whose product is the given index, and extract out of the given number a root answering to one of these indices; and then out of this root extract a root, answering to another of the indices, and so on to the last.

Thus, the fourth root = square root of the square root.

The sixth root = square root of the cube root, &c.

The proof of all roots is by involution.

3. To the remainder bring down the first figure in the next period, and call it the *dividend.*

4. Involve the root to the next inferior power to that, which is given, and multiply it by the number denoting the given power for a *divisor*.

5. Find how many times the divisor may be had in the dividend, and the quotient will be another figure of the root.

6. Involve the whole root to the given power, and subtract it from the given number as before.

7. Bring down the first figure of the next period to the remainder for a new dividend, to which find a new divisor, and so on, till the whole be finished.

EXAMPLES.

1. What is the cube root of 53157376?

$$\begin{array}{rl} & 53\dot{1}57\dot{3}7\dot{6}(376 \\ & \underline{27=3^3} \\ 3^2\times3=27) & 261 \text{ dividend.} \\ & \underline{50653=37^3} \\ 37^2\times3=4107) & 25043 \text{ second dividend.} \\ & \underline{53157376} \\ & 0 \end{array}$$

The following theorems may sometimes be found useful in extracting the root of a vulgar fraction; $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{ab}}{b}=\frac{a}{\sqrt{ab}}$; or universally, $\left.\frac{a}{b}\right|^{\frac{1}{n}}=\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{\left.\overline{ab^{n-1}}\right|^{\frac{1}{n}}}{b}=\frac{a}{\left.\overline{ba^{n-1}}\right|^{\frac{1}{n}}}$.

2. What is the biquadrate root of 19987173376?

Ans. 376.

3. Extract the sursolid, or fifth root, of 30768282110671 5625. Ans. 3145.

4. Extract the square cubed, or sixth root, of 43572838 1009267809889764416. Ans. 27534.

5. Find the seventh root of 34487717467307513182492 153794673. Ans. 32017.

6. Find the eighth root of 112101628132047623624649 7942460481. Ans. 13527.

TO EXTRACT ANY ROOT WHATEVER BY APPROXIMATION.

RULE.

1. Assume the root nearly, and raise it to the same power with the given number, which call the *assumed power*.

2. Then, as the sum of the assumed power multiplied by the index more 1 and the given number multiplied by the index less 1, is to the sum of the given number multiplied by the index more 1 and the assumed power multiplied by the index less 1, so is the assumed root to the required root.

Or, as half the first sum is to the difference between the given and assumed powers, so is the assumed root to the difference between the true and assumed roots; which difference, added or subtracted, gives the true root nearly.

And the operation may be repeated as often as we please by using always the last found root for the assumed root, and its power as aforesaid for the assumed power.

EXAMPLES.

1. Required the fifth root of 21035·8.

Here it appears, that the fifth root is between 7·3 and 7·4, 7·3 being taken, its fifth power is 20730·71593. Hence then

```
21035·8    = given number.
20730·716  = assumed power.
----------
  305·084  = difference.

5 = index.        20730·716      21035·8
5+1=6                     3            2
5—1=4             ---------      -------
6÷2=3             62192·148      42071·6
4÷2=2             42071·6
                  ---------
                 104263·748 = ½ the first sum.

104263·7  :  305·084  ::  7·3  :  ·0213605
                 7·3
              -------
              915252
             2135588
             -------
104263·7)2227·1132(·0213604 = difference,
   ......  208527    7·3
           ------    --------
            14184    7·321360 = root,
            10426       true to the last figure,
            -----
             3758
             3128
             ----
              630
              626
              ---
                4
```

2. What is the third root of 2 ? Ans. 1·259921.

3. What is the sixth root of 21035·8 ? Ans. 5·254037.

4. What is the seventh root of 21035·8 ? Ans. 4·145392.

5. What is the ninth root of 21035·8 ? Ans. 3·022239.

ARITHMETICAL PROGRESSION.

Any rank of numbers, increasing by a common excess or decreasing by a common difference, is said to be in *Arithmetical Progression;* such are the numbers 1, 2, 3, 4, 5, &c. 7, 5, 3, 1; and ·8, ·6, ·4, ·2. When the numbers increase they form an *ascending series;* but when they decrease, they form a *descending series.*

The numbers, which form the series, are called the *terms* of the progression.

Any *three* of the *five* following terms being given, the other two may be readily found.

1. The first term, } commonly called the
2. The last term, } *extremes.*
3. The number of terms.
4. The common difference.
5. The sum of all the terms.

PROBLEM 1.

The first term, the last term, and the number of terms being given, to find the sum of all the terms.

RULE.*

Multiply the sum of the extremes by the number of terms, and half the product will be the answer.

EXAMPLES.

1. The first term of an Arithmetical Progression is 2, the last term 53, and the number of terms 18; required the sum of the series.

$$\begin{array}{r} 53 \\ 2 \\ \hline 55 \\ 18 \\ \hline 440 \\ 55 \\ \hline 2)990 \\ \hline 495 \end{array}$$

Or,

$$\frac{\overline{53+2}\times 18}{2}=495 \text{ the answer.}$$

---

* Suppose another series of the same kind with the given one be placed under it in an inverse order; then will the sum of every two corresponding terms be the same as that of the first and last; consequently any one of those sums, multiplied by the number of terms, must give the whole sum of the two series, and half that sum will evidently be the sum of the given series: thus,

Let 1, 2, 3, 4, 5, 6, 7, be the given series;

and 7, 6, 5, 4, 3, 2, 1, the same inverted;

then $8+8+8+8+8+8+8=8\times7=56$ and $1+2+3+4+5+6+7=\frac{56}{2}=28$. Q. E. D.

2. The first term is 1, the last term 21, and the number of terms 11; required the sum of the series. Ans. 121.

3. How many strokes do the clocks of Venice, which go to 24 o'clock, strike in the compass of a day?

Ans. 300.

4. If 100 stones be placed in a right line, exactly a yard asunder, and the first a yard from a basket, what length of ground will that man go, who gathers them up singly, returning with them one by one to the basket?

Ans. 5 miles and 1300 yards.

PROBLEM 2.

The first term, the last term, and the number of terms being given, to find the common difference.

RULE. 3.*

Divide the difference of the extremes by the number of terms less 1, and the quotient will be the common difference sought.

EXAMPLES.

1. The extremes are 2 and 53, and the number of terms is 18; required the common difference.

---

* The difference of the first and last terms evidently shows the increase of the first term by all the subsequent additions, till it becomes equal to the last; and as the number of those additions is evidently one less than the number of terms, and the increase by every addition equal, it is plain, that the total increase, divided by the number of additions, must give the difference at every one separately; whence the rule is manifest.

```
    53          18
     2           1
   ---         ---
17)51(3         17
   51
   ---
```

Or,

$$\frac{53-2}{18-1}=\frac{51}{17}=3 \text{ the answer.}$$

2. If the extremes be 3 and 19, and the number of terms 9, it is required to find the common difference, and the sum of the whole series.

Ans. The difference is 2, and the sum is 99.

3. A man is to travel from London to a certain place in 12 days, and to go but three miles the first day, increasing every day by an equal excess, so that the last day's journey may be 58 miles; required the daily increase, and the distance of the place from London.

Ans. Daily increase 5, distance 366 miles.

PROBLEM 3.

Given the first term, the last term, and the common difference, to find the number of terms.

RULE.*

Divide the difference of the extremes by the common dif-

---

* By the last problem, the difference of the extremes, divided by the number of terms less 1, gives the common difference; consequently the same, divided by the common difference, must give the number of terms less 1; hence this quotient, augmented by 1, must be the answer to the question.

ference, and the quotient, increased by 1, is the number of terms required.

EXAMPLES.

1. The extremes are 2 and 53, and the common difference 3; what is the number of terms?

$$\begin{array}{r} 53 \\ 2 \\ \hline 3)51 \\ \hline 17 \\ 1 \\ \hline 18 \end{array}$$

Or,

$\frac{53-2}{3}+1=18$ the answer.

In any Arithmetical Progression, the sum of any two of its terms is equal to the sum of any other two terms, taken at an equal distance on contrary sides of the former; or the double of any one term is equal to the sum of any two terms, taken at an equal distance from it on each side.

The sum of any number of terms (n) of the arithmetical series of odd numbers 1, 3, 5, 7, 9, &c. is equal to the square (n^2) of that number.

That is, if 1, 3, 5, 7, 9, &c. be the numbers,

Then will 1, 2^2, 3^2, 4^2, 5^2, &c. be the sums of 1, 2, 3, &c. of those terms.

For, $0+1$ or the sum of 1 term $=1^2$ or 1
$1+3$ or the sum of 2 terms $=2^2$ or 4
$4+5$ or the sum of 3 terms $=3^2$ or 9
$9+7$ or the sum of 4 terms $=4^2$ or 16, &c.

Whence it is plain, that, let n be any number whatsoever, the sum of n terms will be n^2.

The following table contains a summary of the whole doctrine of Arithmetical Progression.

2. If the extremes be 3 and 19, and the common difference 2, what is the number of terms? Ans. 9.

3. A man, going a journey, travelled the first day 5 miles, the last day 35 miles, and increased his journey every day

Cases of *Arithmetical Progression*.

Case	Giv.	Req.	Solution.
1	*adn*	*l*	$\overline{n-1}\times d+a.$
		s	$n\times\overline{a+\overline{n-1}\times\frac{d}{2}}$
2	*adl*	*n*	$\frac{l-a}{d}+1.$
		s	$\frac{\overline{l+a}\times\overline{l-a+d}}{2d}.$
3	*ads*	*n*	$\frac{\sqrt{\overline{2a-d}\vert^2+8ds}-\overline{2a-d}}{2d}.$
		l	$\frac{\sqrt{\overline{2a-d}\vert^2+8ds}-d}{2}.$
4	*als*	*d*	$\frac{\overline{l+a}\times\overline{l-a}}{2s-\overline{l+a}}.$
		n	$\frac{2s}{a+l}.$
5	*ans*	*d*	$\frac{2\times\overline{s-an}}{\overline{n-1}\times n}$
		l	$\frac{2s}{n}-a.$

by 3 miles; how many days did he travel? Ans. 11 days.

Case	Giv.	Req.	Solution.
6	*aln*	*d*	$\frac{l-a}{n-1}.$
		s	$\frac{\overline{a+l}\times n}{2}.$
7	*dnl*	*a*	$l-\overline{n-1}\times d.$
		s	$n\times\overline{l-\overline{n-1}\times\frac{d}{2}}.$
8	*snd*	*a*	$\frac{s}{n}-\frac{d\times\overline{n-1}}{2}.$
		l	$\frac{s}{n}-\frac{d\times\overline{n-1}}{2}.$
9	*dls*	*a*	$\frac{d\pm\sqrt{\overline{2l+d}\vert^{2}-8ds}}{2}.$
		n	$\frac{2l+d\mp\sqrt{\overline{2l+d}^{2}-8ds}}{2d}.$
10	*lns*	*a*	$\frac{2s}{n}-l.$
		d	$\frac{2\times\overline{nl-s}}{\overline{n-1}\times n}.$

Here:
- a = least term.
- n = number of terms.
- s = sum of all the terms.
- d = common difference.
- l = greatest term.

GEOMETRICAL PROGRESSION.

Any series of numbers, the terms of which gradully increase or decrease by a constant multiplication or division, is said to be in *Geometrical Progression.* Thus, 4, 8, 16, 32, 64, &c. and 81, 27, 9, 3, 1, &c. are series in Geometrical Progression, the one increasing by a constant multiplication by 2, and the other decreasing by a constant division by 3.

The number, by which the series is constantly increased or diminished, is called the *ratio.*

PROBLEM 1.

Given the first term, the last term, and the ratio, to find the sum of the series.

RULE.*

Multiply the last term by the ratio, and from the product subtract the first term, and the remainder, divided by the ratio less 1, will give the sum of the series.

---

* Demonstration. Take any series whatever, as 1, 3, 9, 27, 81, 243, &c. multiply this by the ratio, and it will produce the series 3, 9, 27, 81, 243, 729, &c. Now let the sum of the proposed series be what it will, it is plain, that the sum of the second series will be as many times the former sum, as is expressed by the ratio; subtract the first series from the second, and it will give 729—1; which is evidently as many times the sum of the first series, as is expressed by the ratio less 1; consequently $\frac{729-1}{3-1}$ = sum of the proposed series, and is the rule; or 729 is the last term multiplied by the ratio, 1 is the first term, and 3—1 is the ratio less one; and the same will hold, let the series be what it will. Q. E. D.

EXAMPLES.

1. The first term of a series in Geometrical Progression is 1, the last term is 2187, and the ratio 3; what is the sum of the series?

$$\begin{array}{r} 2187 \\ 3 \\ \hline 6561 \\ 1 \\ \hline 3-1=2)6560 \\ \hline 3280 \end{array}$$

Or,

$$\frac{3\times 2187-1}{3-1}=3280 \text{ the answer.}$$

NOTE 1. Since, in any geometrical series of progression, when it consists of four terms, the product of the extremes is equal to the product of the means; and when it consists of three, the product of the extremes is equal to the square of the mean; it follows, that in any geometrical series, when it consists of an even number of terms, the product of the extremes is equal to the product of any two means, equally distant from the extremes; and when the number of terms is odd, the product of the extremes is equal to the square of the mean or middle term, or to the product of any two terms, equally distant from them.

NOTE 2. If $a : b :: c : d$ directly,

Then
$a : c :: b : d$ by alteration.
$b : a :: d : c$ by inversion.
$a+b : b :: c+d : d$ by composition.
$a-b : b :: c-d : d$ by division.
$a : a+b :: c : c+d$ by conversion.
$a+b : a-b :: c+d : c-d$ mixedly.

For in each of these proportions the product of the extremes is equal to that of the means.

2. The extremes of a geometrical progression are 1 and 65536, and the ratio 4; what is the sum of the series?

Ans. 87381.

3. The extremes of a geometrical series are 1024 and 59049, and the ratio is $1\frac{1}{8}$; what is the sum of the series?

Ans. 175099.

PROBLEM 2.

Given the first term and the ratio, to find any other term assigned.

RULE.*

1. Write a few of the leading terms of the series, and place their indices over them, beginning with a cypher.

2. Add together the most convenient indices to make an index less by 1 than the number, expressing the place of the term sought.

3. Multiply the terms of the geometrical series together, belonging to those indices, and make the product a dividend.

---

* DEMONSTRATION. In example 1, where the first term is equal to the ratio, the reason of the rule is evident; for as every term is some power of the ratio, and the indices point out the number of factors, it is plain from the nature of multiplication, that the product of any two terms will be another term corresponding with the index, which is the sum of the indices standing over those respective terms,

And in the second example, where the series does not begin with the ratio, it appears, that every term after the two first contains some power of the ratio, multiplied into the first term, and therefore the rule, in this case, is equally evident.

4. Raise the first term to a power, whose index is 1 less than the number of terms multiplied, and make the result a divisor.

The following table contains all the possible cases of Geometrical Progression.

CASES OF *GEOMETRICAL PROGRESSION.*			
Case	Giv.	Req.	Solution.
1	*arn*	*l*	ar^{n-1}.
		s	$\frac{r^{n-1}}{r-1}\times a$.
2	*arl*	*s*	$l+\frac{l-a}{r-1}$.
		n	$\frac{L,l-L,a}{L,r}+1$.
3	*ars*	*l*	$\frac{\overline{r-1}\times s+a}{r}$.
		n	$\frac{L,\overline{\overline{r-1}\times s+a}-L,a}{L,r}$.
4	*asl*	*r*	$\frac{s-a}{s-l}$.
		n	$\frac{L,l-L,a}{L,\overline{s-a}-L,\overline{s-l}}+1$.

5. Divide the dividend by the divisor, and the quotient will be the term sought.

NOTE. When the first term of the series is equal to the ratio, the indices must begin with an unit, and the indices added must make the entire index of the term required; and

Case	Giv.	Req.	Solution.
5	*ans*	*r*	$r^n - \frac{rs}{a} = \frac{a-s}{a}$.
		l	$l \times \overline{s-l}\vert^{n-1} = a \times \overline{s-a}\vert^{n-1}$.
6	*anl*	*r*	$\overline{\frac{l}{a}}\vert^{\frac{1}{n-1}}$.
		s	$l + \frac{l-a}{\overline{\frac{l}{a}}\vert^{\frac{1}{n-1}} - 1}$.
7	*rnl*	*a*	$\frac{l}{r^{n-1}}$
		s	$l + \frac{l - \frac{l}{r^{n-1}}}{r-1}$.
8	*rns*	*a*	$\frac{r-1}{r^n-1} \times s$.
		l	$\frac{r^n - r^{n-1}}{r^n-1} \times s$.

the product of the different terms, found as before, will give the term required.

EXAMPLES.

1. The first term of a geometrical series is 2, the number of terms 13, and the ratio 2; required the last term.

1, 2, 3, 4, 5, indices.
2, 4, 8, 16, 32, leading terms.
Then 4+4+3+2 = index to the 13th term.
And 16×16×8+4 = 8192 the answer.

In this example the indices must begin with 1, and such of them be chosen, as will make up the entire index to the term required.

Case	Giv.	Req.	Solution.
9	rls	a	$s-r\times\overline{s-l}$.
		n	$\dfrac{L,l-L,\overline{s-r\times\overline{s-l}}}{L,r}+1$.
10	nls	a	$a\times\overline{s-a}\vert^{n-1}=l\times\overline{s-l}\vert^{n-1}$.
		r	$r^n+\dfrac{s}{l-s}r^{n-1}=\dfrac{l}{l-s}$.

Here: a = least term.
l = greatest term.
s = sum of all the terms.
n = number of terms.
r = ratio.
L = Logarithms.

2. Required the 12th term of a geometrical series, whose first term is 3, and ratio 2.

0, 1, 2, 3, 4, 5, 6, indices.
3, 6, 12, 24, 48, 96, 192, leading terms.
Then 6+5= index to the 12th term.
And 192×96=18432 = dividend.

The number of terms multiplied is 2, and 2—1=1 is the power, to which the term 3 is to be raised; but the first power of 3 is 3, and therefore 18432÷3=6144 the 12th term required.

3. The first term of a geometrical series is 1, the ratio 2, and the number of terms 23; required the last term.

Ans. 4194304.

QUESTIONS

TO BE SOLVED BY THE TWO PRECEDING PROBLEMS.

1. A person being asked to dispose of a fine horse, said he would sell him on condition of having one farthing for the first nail in his shoes, 2 farthings for the second, one penny for the third, and so on, doubling the price of every nail to 32, the number of nails in his four shoes; what would the horse be sold for at that rate?

Ans. 4473924l. 5s. $3\frac{3}{4}$d.

2. A young man, skilled in numbers, agreed with a farmer to work for him eleven years without any other reward than the produce of one wheat corn for the first year, and that produce to be sowed the second year, and so on from year to year till the end of the time, allowing the increase to be in a ten fold proportion; what quantity of wheat is due for

such service, and to what does it amount at a dollar per bushel?

Ans. 226056$\frac{1}{8}$ bushels, allowing 7680 wheat corns to be a pint; and the amount is 226056$\frac{1}{8}$ dollars.

3. What debt will be discharged in a year, or twelve months, by paying $ 1 the first month, $ 2 the second, $ 4 the third, and so on, each succeeding payment being double the last; and what will the last payment be?

Ans. The debt is $ 4095 and the last payment $ 2048.

SIMPLE INTEREST.

INTEREST is the premium, allowed for the loan of money.

The sum, which is lent, is called the *principal.*

The sum of the principal and interest is called the *amount.*

Interest is allowed at so much *per cent. per annum*, which premium *per cent. per annum*, or interest of 100l. for a year, is called the *rate* of interest.

Interest is of two sorts, *simple* and *compound.*

Simple Interest is that, which is allowed only for the principal lent.

NOTE. Commission, Brokerage, Insurance, Stocks,* and, in general, whatever is at a certain rate, or sum per cent. are calculated like Simple Interest.

---

* *Stock* is a general name for public funds, and capitals of trading companies, the shares of which are transferable from one person to another.

RULE.*

1. Multiply the principal by the rate, and divide the product by 100; and the quotient is the answer for one year.

2. Multiply the interest for one year by the given number of years, and the product is the answer for that time.

3. If there be parts of a year, as months or days, work for the months by the aliquot parts of a year, and for the days by Simple Proportion.

EXAMPLES.

1. What is the interest of 450l. for a year, at 5 per cent. per annum?

```
      450l.
          5
    -------
1,00)22·50
         20
    -------
      10·00 Ans. 22l. and 50/100 = 5/10 = ·5 = 10s.
```

2. What is the interest of 720l. for 3 years, at 5 per cent. per annum;

```
 720l.        36
     5         3
 -----      ----
 36·00      108l. Ans.
```

3. What is the interest of 107l. for 117 days, at 4¾ per cent. per annum?

```
 107l.    5  1 7 3·2    5  1 7 3·2
    4¾           11              7
 -----    ----------    ----------
```

---

* The rule is evidently an application of Simple Proportion and Practice.

	428	55 18 1 3·2	35 11 6 2·4
	53 10	10	
	26 15		
11×10+7=117		559 1 6 0	
	5·08 5	35 11 6 2·4	
	20		
		365)594 13 0 2·4(1l. 12s. $7\frac{8}{1825}$d,	the answer.
	1·65	365	
	12		
		229	
	7·80	20	
	4		
		)4593	
	3·20	365	
q. q. $2·4=2\frac{2}{5}=\frac{3}{5}$d.		943	
		730	
$1\frac{3}{5}=\frac{8}{5}$			
		213	
$\frac{\frac{8}{5}}{365}=\frac{8}{1825}$		12	
		)2556	
		2555	
		$1\frac{3}{5}$	

4. What is the interest of $607·50 for 5 years, at 6 per cent. per annum? Ans. $182·25.

5. What is the interest of 213l. from Feb. 12, to June 5, 1796, it being leap year, at $3\frac{1}{2}$ per cent. per annum?

Ans. 2l. 6s. 6d. $3\frac{601}{1825}$q.

SIMPLE INTEREST BY DECIMALS.

RULE.*

Multiply continually the principal, ratio, and time, and it will give the interest required.

---

* The following theorems will show all the possible cases of Simple Interest, where p = principal, t = time, r = ratio, and a = amount.

Ratio is the simple interest of 1l. for 1 year, at the rate per cent. agreed on; thus the ratio

at	3 per cent. is	·03.
	$3\frac{1}{2}$	·035.
	4	·04.
	$4\frac{1}{2}$	·045.
	5	·05.
	$5\frac{1}{2}$	·055.
	6	·06.

EXAMPLES.

1. What is the interest of 945l. 10s. for 3 years, at 5 per cent. per annum?

```
 945·5
   ·05
 ------
 47·275
      3
 -------
141·825
     20
 -------
 16·500
     12
 ------
  6·000      Ans. 141l. 16s. 6d.
```

2. What is the interest of 796l. 15s. for 5 years, at $4\frac{1}{2}$ per cent. per annum? Ans. 179l. 5s. $4\frac{1}{2}$d.

3. What is the interest of 537l. 15s. from November 11, 1764, to June 5, 1765, at $3\frac{5}{8}$ per cent.? Ans. 11l. $\frac{1}{4}$d.

I. $ptr+p=a.$ II. $\frac{a-p}{rp}=t.$

III. $\frac{a}{tr+1}=p$ IV. $\frac{a-p}{tp}=r.$

COMMISSION.

Commission is an allowance of so much per cent. to a factor or correspondent abroad, for buying and selling goods for his employer.

EXAMPLES.

1. What comes the commission of 500l. 13s. 6d. to at $3\frac{1}{2}$ per cent.?

500l.	13s.	6d.	
		$3\frac{1}{2}$	
1502	0	6	
250	6	9	
17·52	7	3	
20			
10·47			
12			
5·67			
4			
2·68			Ans. 17l. 10s. $5\frac{1}{2}$d.

2. My correspondent writes me word, that he has bought goods on my account to the value of 754l. 17s. What does his commission come to at $2\frac{1}{2}$ per cent.?

Ans. 18l. 17s. $4\frac{3}{4}$d.

3. What must I allow my correspondent for disbursing on my account 529l. 18s. 5d. at $2\frac{1}{4}$ per cent.?

Ans. 11l. 18s. $5\frac{1}{2}$d.

BROKERAGE.

Brokerage is an allowance of so much per cent. to a person, called a Broker, for assisting merchants or factors in procuring or disposing of goods.

EXAMPLES.

1. What is the brokerage of 610l. at 5s. or $\frac{1}{4}$ per cent.?

5s. is $\frac{1}{4}$	610l.		
	1·52	10	
	20		
	10·50		
	12		
	6·00		Ans. 1l. 10s. 6d.

2. If I allow my broker $3\frac{3}{4}$ per cent. what may he demand, when he sells goods to the value of 876l. 5s. 10d.?

Ans. 32l. 17s. $2\frac{1}{2}$d.

3. What is the brokerage of 879l. 18s. at $\frac{3}{8}$ per cent.?

Ans. 3l. 5s. $11\frac{3}{4}$d.

INSURANCE.

INSURANCE is a premium of so much per cent. given to certain persons and offices for a security of making good the loss of ships, houses, merchandize, &c. which may happen from storms, fire, &c.

EXAMPLES.

1. What is the insurance of 874l. 13s. 6d. at 13½ per cent.?

874l.	13s.	6d.
		12
10496	2	0
874	13	6
437	6	9
118·08	2	3
20		
1·62		
12		
7·47		
4		
1·88	Ans. 118l. 1s. 7½d.	

2. What is the insurance of 900l. at 10¾ per cent.?

Ans. 96l. 15s.

3. What is the insurance of 1200l. at 7⅝ per cent.?

Ans. 91l. 10s.

DISCOUNT.

Discount is an allowance, made for the payment of any sum of money before it becomes due; and is the difference between that sum, due sometime hence, and its present worth.

The *present worth* of any sum, or debt, due some time hence, is such a sum, as, if put to interest, would in that time and at the rate per cent. for which the discount is to be made, amount to the sum or debt then due.

RULE.*

1. As the amount of 100l. for the given rate and time is to 100l. so is the given sum or debt to the present worth.

---

* That an allowance ought to be made for paying money before it becomes due, which is supposed to bear no interest till after it is due, is very reasonable; for, if I keep the money in my own hands till the debt becomes due, it is plain I may make an advantage of it by putting it out to interest for that time; but, if I pay it before it is due, it is giving that benefit to another; therefore we have only to inquire what discount ought to be allowed. And here some debtors will be ready to say, that since by not paying the money till it becomes due, they may employ it at interest, therefore by paying it before due they shall lose that interest, and for that reason all such interest ought to be discounted: but that is false, for they cannot be said to lose that interest till the time, when the debt shall become due; whereas we are to consider what would properly be lost at present, by paying the debt before it becomes due; and this can, in point of equity or justice, be no other than such a sum, as, being put out to interest till the debt becomes due, would amount to the in-

2. Subtract the present worth from the given sum, and the remainder is the discount required.

Or,

As the amount of 100l. for the given rate and time is to the interest of 100l. for that time, so is the given sum or debt to the discount required.

EXAMPLES.

1. What is the discount of 573l. 15s. due 3 years hence, at $4\frac{1}{2}$ per cent.?

---

terest of the debt for the same time. It is beside plain, that the advantage arising from discharging a debt, due sometime hence, by a present paymemt, according to the principles we have mentioned, is exactly the same as employing the whole sum at interest till the time, when the debt becomes due; for, if the discount allowed for present payment be put out to interest for that time, its amount will be the same as the interest of the whole debt for the same time: thus the discount of 105l. due one year hence, reckoning interest at 5 per cent. will be 5l. and 5l. put out to interest at 5 per cent. for one year, will amount to 5l. 5s. which is exactly equal to the interest of 105l. for one year at 5 per cent.

The truth of the rule for working is evident from the nature of Simple Interest; for since the debt may be considcred as the amount of some principal, called here the present worth, at a certain rate per cent. and for the given time, that amount must be in the same proportion, either to its principal or interest, as the amount of any other sum, at the same rate, and for the same time, is to its principal or interest.

```
4l. 10s.
      3
---------
 13  10
100
---------      l.   s.        l.   s.
113  10   :   13   10   ::   573  15  l.
 20           20              20
----         ----            ----
2270          270            11475
                               270
                             ------
                             803250
                            22950
                            -------(2,0)
                        227,0)309825,0( 136,4
                             828        ------
                             1472        68  4
                              1105
                               197
                                12
                             -------
                          227)2364(10
                                94
                                 4
                               ----
                               376(1
                               149
```

Ans. 68l. 4s. 10¼d.

2. What is the present worth of 150l. payable in ¼ of a year, discount being at 5 per cent.?

Ans. 148l. 2s. 11½d.

3. Bought a quantity of goods for 150l. ready money, and sold them again for 200l. payable at ¾ of a year hence; what was the gain in ready money, supposing discount to be made at 5 per cent.? Ans. 42l. 15s. 5d.

4. What is the present worth of 120l. payable as follows, viz. 50l. at 3 months, 50l. at 5 months, and the rest at 8 months, discount being at 6 per cent.?

Ans. 117l. 5s. 5½d.

DISCOUNT BY DECIMALS.

RULE.*

As the amount of 1l. for the given time is to 1l. so is the interest of the debt for the said time to the discount required.

Subtract the discount from the principal, and the remainder will be the present worth.

---

* Let m represent any debt, and n the time of payment; then will the following Tables exhibit all the variety, that can happen with respect to present worth and discount.

OF THE *PRESENT WORTH* OF MONEY PAID BEFORE IT IS DUE AT *SIMPLE INTEREST*.			
The present worth of any sum m.			
Rate per cent.	For n years.	n months.	n days.
r per cent.	$\frac{100m}{nr+100}$	$\frac{1200m}{nr+100}$	$\frac{36500m}{nr+36500}$
3 per cent	$\frac{100m}{3n+100}$	$\frac{400m}{n+400}$	$\frac{36500m}{3n+36500}$
4 per cent.	$\frac{25m}{n+25}$	$\frac{300m}{n+300}$	$\frac{9125m}{n+9125}$
5 per cent.	$\frac{20m}{n+20}$	$\frac{240m}{n+240}$	$\frac{7300m}{n+7300}$

EXAMPLES.

What is the discount of 573l. 15s. due 3 years hence, at 4½ per cent. per annum?

$\cdot045\times3+1=1\cdot135$=amount of 1l. for the given time.

And $573\cdot75\times\cdot045\times3=77\cdot45625$= interest of the debt for the given time.

$$1\cdot135 : 1 :: 77\cdot45625 :$$

```
1·135)77·45625(68·243
        6810
        ----
         9356
         9080
         ----
          2762
          2270
          ----
```

Of *Discounts* to be allowed for paying of Money before it becomes due at *Simple Interest*.			
	The discount of any sum *m*.		
Rate per cent.	For *n* years.	*n* months.	*n* days.
r per cent	$\frac{mnr}{nr+100}$	$\frac{mnr}{nr+1200}$	$\frac{mnr}{nr+36500}$
3 per cent.	$\frac{3mn}{3n+100}$	$\frac{mn}{n+400}$	$\frac{3mn}{3n+36500}$
4 per cent.	$\frac{mn}{n+25}$	$\frac{mn}{n+300}$	$\frac{mn}{n+9125}$
5 per cent.	$\frac{mn}{n+20}$	$\frac{mn}{n+240}$	$\frac{mn}{n+7300}$

4925
4540

3850
3405

68·243=68l. 4s. 10¼d. Ans. 445

2. What is the discount of 725l. 16s. for five months, at $3\frac{7}{8}$ per cent. per annum? Ans. 11l. 10s. 7¾d.

3. What ready money will discharge a debt of 1377l. 13s. 4d. due 2 years, 3 quarters and 25 days hence, discounting at $4\frac{3}{8}$ per cent. per annum?

Ans. 1226l. 8s. 8½d.

EQUATION OF PAYMENTS.

Equation of Payments is the finding a time to pay at once several debts, due at different times, so that no loss shall be sustained by either party.

RULE.*

Multiply each payment by the time, at which it is due; then divide the sum of the products by the sum of the payments, and the quotient will be the time required.

* This rule is founded on a supposition, that the sum of the interests of the several debts, which are payable before the equated time, from their terms to that time, ought to be equal to the sum of the interests of the debts payable after the equated time, from that time to their terms. Among others, who defend this principle, Mr. Cocker endeavours to prove it to be right by

EXAMPLES.

1. A owes B 190l. to be paid as follows, viz. 50l. in 6 months, 60l. in 7 months, and 80l. in 10 months; what is the equated time to pay the whole?

---

this argument; that what is gained by keeping some of the debts after they are due, is lost by paying others before they are due. But this cannot be the case; for, though by keeping a debt unpaid after it is due there is gained the interest of it for that time, yet by paying a debt before it is due the payer does not lose the interest for that time, but the discount only, which it less than the interest, and therefore the rule is not true.

Although this rule be not accurately true, yet in most questions, that occur in business, the error is so trifling, that it will be much used.

That the rule is universally agreeable to the supposition may be thus demonstrated.

Let
- d = first debt payable, and the distance of its term of payment t.
- D = last debt payable, and the distance of its term T.
- x = distance of the equated time.
- r = rate of interest of 1l. for one year.

Then, since x lies between T and t:
- The distance of the time t and x is $= x-t$.
- The distance of the time T and x is $= T-x$

Now the interest of d for the time $x-t$ is $\overline{x-t}\times dr$; and the interest of D for the time $T-x$ is $\overline{T-x}\times Dr$; therefore $\overline{x-t}\times dr = \overline{T-x}\times Dr$ by the supposition; and from this equation x is found $= \frac{DT+dt}{D+d}$, which is the rule. And the same might be shown of any number of payments.

The true rule is given in equation of payments by decimals.

$$50\times\ 6=300$$
$$60\times\ 7=420$$
$$80+\ 10=800$$

$$50+60+80=190)1520(8$$
$$1520$$

Ans. 8 months.

2. A owes B 52l. 7s. 6d. to be paid in $4\frac{1}{2}$ months, 80l. 10s. to be paid in $3\frac{1}{2}$ months, and 76l. 2s. 6d. to be paid in 5 months; what is the equated time to pay the whole?

Ans. 4 months, 8 days.

3. A owes B 240l. to be paid in 6 months, but in one month and a half pays him 60l. and in $4\frac{1}{2}$ months after that 80l. more; how much longer than 6 months should B in equity defer the rest? Ans. 2·7 months.

4. A debt is to be paid as follows, viz. $\frac{1}{4}$ at 2 months, $\frac{1}{8}$ at 3 months, $\frac{3}{8}$ at 4 months, $\frac{1}{8}$ at 5 months, and the rest at 7 months; what is the equated time to pay the whole?

Ans. 4 months and 18 days.

EQUATION OF PAYMENTS BY DECIMALS.

Two debts being due at different times, to find the equated time to pay the whole.

RULE.*

1. To the sum of both payments add the continual product of the first payment, the rate, or interest of 1l. for one

* No rule in Arithmetic has been the occasion of so many disputes as that of Equation of Payments. Almost every writer on this subject has endeavoured to show the fallacy of the methods, used by other authors, and to substitute a new one in their stead. But the only true rule seems to be that of Mr.

year, and the time between the payments, and call this the first number.

2. Multiply twice the first payment by the rate, and call this the second number.

MALCOLM, or one similar to it in its essential principles, derived from the consideration of interest and discount.

The rule, given above, is the same as Mr. MALCOLM's, except it is not encumbered with the time before any payment is due, that being no necessary part of the operation.

DEMONSTRATION OF THE RULE. Suppose a sum of money to be due immediately, and another sum at the expiration of a certain given time, and it is proposed to find a time to pay the whole at once, so that neither party shall sustain loss.

Now it is plain, that the equated time must fall between those of the two payments; and that what is got by keeping the first debt after it is due, should be equal to what is lost by paying the second debt before it is due.

But the gain, arising from the keeping of a sum of money after it is due, is evidently equal to the interest of the debt for that time.

And the loss, which is sustained by paying of a sum of money before it is due, is evidently equal to the discount of the debt for that time.

Therefore it is obvious, that the debtor must retain the sum immediately due, or the first payment, till its interest shall be equal to the discount of the second sum for the time it is paid before due; because, in that case, the gain and the loss will be equal, and consequently neither party can be loser.

Now to find such a time, let a = first payment, b = second, and t = time between the payments; r = rate, or interest o

3. Divide the first number by the second, and call the quotient the third number.

---

11. for one year, and x = equated time after the first payment.

Then arx = interest of a for x time,

and $\frac{btr-brx}{1+tr-rx}$ = discount of b for the time $t-x$.

But $arx=\frac{btr-brx}{1+tr-rx}$ by the question, from which equation x is found $=\frac{a+b+atr}{2ar}\pm\overline{\overline{\frac{a+b+atr}{2ar}}\big|^{2}-\frac{bt}{ar}}\Big|^{\frac{1}{2}}$.

Let $\frac{a+b+atr}{2ar}$ be put equal to n, and $\frac{bt}{ar}=m$.

Then it is evident, that n, or its equal $\overline{n^2}\big|^{\frac{1}{2}}$ is greater than $\overline{n^2-m}\big|^{\frac{1}{2}}$, and therefore x will have two affirmative values, the quantities $n+\overline{n^2-m}\big|^{\frac{1}{2}}$ and $n-\overline{n^2-m}\big|^{\frac{1}{2}}$ being both positive.

But only one of those values will answer the conditions of the question; and, in all cases of this problem, x will be $=n-\overline{n^2-m}\big|^{\frac{1}{2}}$.

For suppose the contrary, and let $x=n+\overline{n^2-m}\big|^{\frac{1}{2}}$.

Then $t-x=t-n-\overline{n^2-m}\big|^{\frac{1}{2}}=\overline{\overline{t-n}\big|^{2}}\Big|^{\frac{1}{2}}-\overline{n^2-m}\big|^{\frac{1}{2}}=\overline{t^2-2tn+n^2}\big|^{\frac{1}{2}}-\overline{n^2-m}\big|^{\frac{1}{2}}=\overline{n^2+t^2-2tn}\big|^{\frac{1}{2}}-\overline{n^2-m}\big|^{\frac{1}{2}}$.

Now, since $\overline{a+b+atr}\times\frac{1}{2ar}=n$, and $bt\times\frac{1}{ar}=m$, we shall have from the first of these equations $t^2-2tn=-\overline{bt-at}\times\frac{1}{ar}$,

and consequently $t-x=\overline{n^2-\overline{bt-at}\times\frac{1}{ar}}\Big|^{\frac{1}{2}}-\overline{n^2-bt\times\frac{1}{ar}}\Big|^{\frac{1}{2}}$.

But $\overline{n^2-bt\times\frac{1}{ra}}\Big|^{\frac{1}{2}}$ is evidently greater than $\overline{n^2-\overline{bt-at}\times\frac{1}{ar}}\Big|^{\frac{1}{2}}$,

and therefore $\overline{n^2-bt\times\frac{1}{ar}}\Big|^{\frac{1}{2}}-\overline{n^2-\overline{bt-at}\times\frac{1}{ar}}\Big|^{\frac{1}{2}}$, or its equal $t-x$,

4. Call the square of the third number the fourth number,

5. Divide the product of the second payment, and time between the payments, by the product of the first payment and the rate, and call the quotient the fifth number.

---

must be a negative quantity; and consequently x will be greater than t, that is, the equated time will fall beyond the second payment, which is absurd. The value of x therefore cannot be $= \frac{a+b+atr}{2ar} + \overline{\overline{\frac{a+b+atr}{2ar}}\Big|^2 \quad \frac{bt}{ar}}\Big|^{\frac{1}{2}}$, but must in all cases be

$= \frac{a+b+atr}{2ar} \quad \overline{\overline{\frac{a+b+atr}{2ar}}\Big|^2 - \frac{bt}{ar}}\Big|^{\frac{1}{2}}$, which is the same as the rule.

From this it appears, that the double sign, made use of by Mr. Malcolm, and every author since, who has given his method, cannot obtain, and that there is no ambiguity in the problem.

In like manner it might be shown, that the directions, usually given for finding the equated time, when there are more than two payments, will not agree with the hypothesis; but this may be easily seen by working an example at large, and examining the truth of the conclusion.

The equated time for any number of payments may be readily found when the question is proposed in numbers, but it would not be easy to give algebraic theorems for those cases, on account of the variation of the debts and times, and the difficulty of finding between which of the payments the equated time would happen.

Supposing r to be the amount of 1l. for one year, and the other letters as before, then $t - \frac{\frac{log.\,ar+b}{a}}{log.\,r}$ will be a general theorem for the equated time of any two payments, reckoning compound interest, and is found in the same manner as the former.

6. From the fourth number take the fifth, and call the square root of the difference the sixth number.

7. Then the difference of the third and sixth numbers is the equated time, after the first payment is due.

EXAMPLES.

1. There is 100l. payable one year hence, and 105l. payable 3 years hence; what is the equated time, allowing simple interest at 5 per cent. per annum?

```
     100          100
     ·05            2
    ─────        ─────
    5·00          200
       2          ·05
   ──────       ──────
   10·00         10·00=2d number.
  100
  105
 ───────
10)215=1st number.
 ─────
  21·5=3d number.
  21·5
 ─────
 1075
 215
430
 ───────
462·25=4th number.        105
                            2
                         ────
    1st payment × rate = 5)210
                         ────
                          42=5th number.
```

```
   462·25          21·5
    42             (20·5=6th number.
   ------          ----
   . . .            1= equated time from the first
   420·25(20·5         payment, and ∴ 2 years =
   4                   whole equated time.
   ------
405)2025
    2025
    ----
```

2. Suppose 400l. are to be paid at the end of 2 years, and 2100l. at the end of 8 years; what is the equated time for one payment, reckoning 5 per cent. simple interest?

Ans. 7 years.

3. Suppose 300l. are to be paid at one year's end, and 300l. more at the end of $1\frac{1}{2}$ year; it is required to find the time to pay it at one payment, 4 per cent. simple interest being allowed?

Ans. 1·248437 year.

COMPOUND INTEREST.

Compound Interest is that, which arises from the principal and interest taken together, as it becomes due, at the end of each stated time of payment.

RULE.*

1. Find the amount of the given principal for the time of the first payment by Simple Interest.

2. Consider this amount as the principal for the second

* The reason of this rule is evident from the definition, and the principles of Simple Interest.

payment, whose amount calculate as before, and so on through all the payments to the last, still accounting the last amount as the principal for the next payment.

EXAMPLES.

1. What is the amount of 320l. 10s. for 4 years, at 5 per cent. per annum, compound interest?

$\frac{1}{20}$)320l.	10s.		1st year's principal.
16	0	6d.	1st year's interest.
$\frac{1}{20}$)336	10	9	2d year's principal.
16	16	$6\frac{1}{4}$	2d year's interest.
$\frac{1}{20}$)353	7	$0\frac{1}{3}$	3d year's principal.
17	13	4	3d year's interest.
$\frac{1}{20}$)371	0	$4\frac{1}{4}$	4th year's principal.
18	11	0	4th year's interest.
389	11	$4\frac{1}{4}$	whole amount, or answer required.

2. What is the compound interest of 760l. 10s. forborn 4 years, at 4 per cent.? Ans. 129l. 3s. $6\frac{1}{4}$d.

3. What is the compound interest of 410l. forborn for $2\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent. per annum; interest payable half-yearly? Ans. 48l. 4s. $11\frac{3}{4}$d.

4. Find the several amounts of 50l. payable yearly, half-yearly, and quarterly, being forborn 5 years, at 5 per cent. per annum, compound interest.

Ans. 63l. 16s. $3\frac{1}{4}$d. 64l. and 64l. 1s. $9\frac{1}{2}$d.

COMPOUND INTEREST BY DECIMALS.

RULE.*

1. Find the amount of 1l. for one year at the given rate per cent.

---

* DEMONSTRATION. Let r = amount of 1l. for one year, and p = principal or given sum; then since r is the amount of 1l. for one year, r^2 will be its amount for 2 years, r^3 for 3 years, and so on; for when the rate and time are the same, all principal sums are necessarily as their amounts; and consequently as r is the principal for the second year, it will be as $1 : r :: r : r^2$ = amount for the second year, or principal for the third; and again, as $1 : r :: r^2 : r^3$ = amount for the third year, or principal for the fourth, and so on to any number of years. And if the number of years be denoted by t, the amount of 1l. for t years will be r^t. Hence it will appear, that the amount of any other principal sum p for t years is pr^t; for as $1 : r^t :: p : pr^t$, the same as in the rule.

If the rate of interest be determined to any other time than a year, as $\frac{1}{2}$, $\frac{1}{4}$, &c. the rule is the same, and then t will represent that stated time.

Let:
- r = amount of 1l. for one year at the given rate per cent.
- p = principal or sum at interest.
- i = interest.
- t = time.
- m = amount for the time t.

Then the following theorems will exhibit the solutions of all the cases in compound interest.

2. Involve the amount, thus found, to such a power, as is denoted by the number of years.

I. $pr^t = m$ II. $pr^t - p = i$.

III. $\frac{m}{r^t} = p$ IV. $\left.\frac{m}{p}\right|^{\frac{1}{t}} = r$.

The most convenient way of giving the theorem for the *time*, as well as for all the other cases, will be by logarithms, as follows.

I. $t \times log.\ r + log.\ p = log.\ m$. II. $log.\ m - t \times log.\ r = log.\ p$.

III. $\frac{log.\ m - log.\ p}{log.\ r} = t$. IV. $\frac{log.\ m - log.\ p}{t} = log.\ r$.

If the compound interest, or amount of any sum, be required for the parts of a year, it may be determined as follows.

I. *When the time is any aliquot part of a year.*

RULE.

1. Find the amount of 1l. for one year, as before, and that root of it, which is denoted by the aliquot part, will be the amount sought.

2. Multiply the amount, thus found, by the principal, and it will be the amount of the given sum required.

II. *When the time is not an aliquot part of a year.*

RULE.

1. Reduce the time into days, and the 365th root of the amount of 1l. for one year is the amount for one day.

2. Raise this amount to that power, whose index is equal to the number of days, and it will be the amount of 1l. for the given time.

3. Multiply this power by the principal, or given sum, and the product will be the amount required.

4. Subtract the principal from the amount, and the remainder will be the interest.

EXAMPLES.

1. What is the compound interest of 500l. for 4 years, at 5 per cent. per annum ?

```
            1·05 = amount of 1l. for one year at 5 per
            1·05     cent.
            ----
             525
           1050
           ------
          1·1025
           11025
         --------
           55125
          22050
        110250
       11525
       -----------
   1.21550625=4th power of 1·05.
            500=principal.
   --------------
   607·75312500= amount.
   500
   --------------
   107·753125=107l. 15s. 0¾d. = interest required.
```

---

3. Multiply this amount by the principal, and it will be the amount of the given sum required.

To avoid extracting very high roots, the same may be done by logarithms thus ; divide the logarithm of the rate, or amount of 1l. for one year, by the denominator of the given aliquot part, and the quotient will be the logarithm of the root sought.

2. What is the amount of 760l. 10s. for 4 years, at 4 per cent.? Ans. 889l. 13s. 6¾d.

3. What is the mount of 721l. for 21 years, at 4 per cent. per annum? Ans. 1642l. 19s. 10d.

4. What is the amount of 217l. forborn 2¼ years, at 5 per cent. per annum, supposing the interest payable quarterly? Ans. 242l. 13s. 4½d.

ANNUITIES.

An Annuity is a sum of money payable every year, for a certain number of years, or forever.

When the debtor keeps the annuity in his own hands, beyond the time of payment, it is said to be in *arrears*.

The sum of all the annuities for the time they have been forborn, together with the interest due upon each, is called the *amount*.

If an annuity be to be bought off, or paid all at once, at the beginning of the first year, the price, which ought to be given for it, is called the *present worth*.

To find the amount of an Annuity at Simple Interest.

RULE.*

1. Find the sum of the natural series of numbers 1, 2, 3, &c. to the number of years less one.

---

* Demonstration. Whatever the time is, there is due upon the first year's annuity, as many years' interest as the whole

2. Multiply this sum by one year's interest of the annuity, and the product will be the whole interest due upon the annuity.

3. To this product add the product of the annuity and time, and the product will be the amount sought.

---

number of years less one; and gradually one less upon every succeeding year to the last but one; upon which there is due only one year's interest, and none upon the last; therefore in the whole there is due as many years' interest of the annuity, as the sum of the series 1, 2, 3, 4, &c. to the number of years less one. Consequently one year's interest, multiplied by this sum, must be the whole interest due: to which, if all the annuities be added, the sum is plainly the amount. Q. E. D.

Let r be the ratio, n the annuity, t the time, and a the amount.

Then let the following theorems give the solutions of all the different cases.

I. $\frac{t^2rn - trn}{2} + tn = a$ II. $\frac{2a - 2tn}{t^2n - tn} = r.$

III. $\frac{2}{t^2r - tr + 2t} = n.$ IV. $\overline{\frac{2a}{rn} + \frac{d}{4}}\Big|^{\frac{1}{2}} - \frac{d}{2} = t.$

In the last theorem $d = \frac{2n - rn}{rn}$, and in theorem first, if a sum cannot be found equal to the amount, the problem is impossible in whole years.

Note. Some writers look upon this method of finding the amount of an annuity as a species of *Compound Interest*; the annuity itself, they say, being properly the Simple Interest, and the capital, whence it arises, the principal.

NOTE. When the annuity is to be paid half-yearly or quarterly; then take, in the former case, ½ the ratio, ½ the annuity, and twice the number of years; and in the latter case, ¼ the ratio, ¼ the annuity, and 4 times the number of years, and proceed as before.

EXAMPLES.

1. What is the amount of an annuity of 50l. for 7 years, allowing simple interest at 5 per cent.?

1+2+3+4+5+6=21=3×7

2l. 10s.=1 year's interest of 50l.

3

7 10

7

52 10

350 0=50l. × 7

402l. 10s. = amount required.

2. If a pension of 600l. per annum be forborn 5 years, what will it amount to, allowing 4 per cent. simple interest?

Ans. 3240l.

3. What will an annuity of 250l. amount to in 7 years, to be paid by half yearly payments, at 6 per cent. per annum, simple interest?

Ans. 2091l. 5s.

To find the present worth of an Annuity at Simple Interest.

RULE.*

Find the present worth of each year by itself, discounting from the time it becomes due, and the sum of all these will be the present worth required.

---

* The reason of this rule is manifest from the nature of discount, for all the annuities may be considered separately, as so many single and independent debts, due after 1, 2, 3, &c. years; so that the present worth of each being found, their sum must be the present worth of the whole.

The estimation, however, of annuities at simple interest is highly unreasonable and absurd. One instance only will be sufficient to show the truth of this assertion. The price of an annuity of 50l. to continue 40 years, discounting at 5 per cent. will, by either of the rules, amount to a sum, of which one year's interest only exceeds the annuity. Would it not therefore be highly ridiculous to give, for an annuity to continue only 40 years, a sum, which would yield a greater yearly interest forever.

It is most equitable to allow compound interst.

Let p = present worth, and the other letters as before.

$$\text{Then}\begin{cases} n\times\overline{\dfrac{1}{1\times r}+\dfrac{1}{1+2r}+\dfrac{1}{1+3r},\ \&\text{c. to } \dfrac{1}{1+tr}}=p. \\ p\div\overline{\dfrac{1}{1+r}+\dfrac{1}{1+2r}+\dfrac{1}{1+3r},\ \&\text{c. to } \dfrac{1}{1+tr}}=n. \end{cases}$$

The other two theorems for the time and rate cannot be given in general terms.

EXAMPLES.

1. What is the present worth of an annuity of 100l. to continue 5 years, at 6 per cent. per annum, simple interest?

106	:	100	: :	100	:	94·3396	=	present worth the first year.
112	:	100	: :	100	:	89·2857	=	2d year.
118	:	100	: :	100	:	84·7457	=	3d year.
124	:	100	: :	100	:	80·6451	=	4th year.
130	:	100	: :	100	:	76·9230	=	5th year.

425.9391 = 425l. 18s. 9¾d. = present worth of the annuity required.

2. What is the present worth of an annuity or pension of 500l. to continue 4 years, at 5 per cent. per annum, simple interest? Ans. 1782l. 3s. 8½d.

To find the Amount of an Annuity at Compound Interest.

RULE.*

Make 1 the first term of a geometrical progression, and the amount of 1l. for 1 year, at the given rate per cent. the ratio.

---

* Demonstration. It is plain, that upon the first year's annuity, there may be due as many year's compound interest as the given number of years less one, and gradually one year's interest less upon every succeeding year to that preceding the last, which has but one year's interest, and the last bears no interest.

Let r therefore = rate, or amount of 1l. for 1 year; then the series of amounts of 1l. annuity, for several year's, from the

2. Carry the series to as many terms as the number of years, and find its sum.

3. Multiply the sum thus found by the given annuity, and the product will be the amount sought.

EXAMPLES.

1. What is the amount of an annuity of 40l. to continue 5 years, allowing 5 per cent. compound interest?

---

first to the last, is 1, r, r^2, r^3, &c. to r^{t-1}. And the sum of this according to the rule in Geometrical Progression, will be $\frac{r^t-1}{r-1}=$ amount of 1l. annuity for t years. And all annuities are proportional to their amounts, therefore $1 : \frac{r^t-1}{r-1} :: n : \frac{r^t-1}{r-1}\times n =$ amount of any given annuity n. Q. E. D.

Let $r =$ rate, or amount of 1l. for one year, and the other letters as before, then $\frac{r^t-1}{r-1}\times n=a$, and $\frac{ar-a}{r^t-1}=n$.

And from these equations all the cases relating to annuities, or pensions in arrears, may be conveniently exhibited in logarithmic terms, thus:

I. $Log.\ n+Log.\ \overline{r^t-1}-Log.\ \overline{r-1}=Log.\ a.$

II. $Log.\ a-Log.\ \overline{r^t-1}+Log.\ \overline{r-1}=Log.\ n.$

III. $\frac{Log.\ \overline{ar-a+n}-Log.\ n}{Log.\ r.}=t.$ IV. $r^t-\frac{ar}{n}+\frac{a}{n}-1=0.$

$$1+1.05+\overline{1.05}|^2+\overline{1.05}|^3+\overline{1.05}|^4=5.52563125$$

```
5.52563125
        40
----------
221.02525
       20
---------
   0.505
      12
   -----
   6.06     Ans. 221l. 6d.
```

2. If 50l. yearly rent, or annuity, be forborn 7 years, what will it amount to, at 4 per cent. per annum, compound interest? Ans. 394l. 18s. 3½d.

To find the present value of Annuities at Compound Interest.

RULE.*

1. Divide the annuity by the ratio, or the amount of 1l. for one year, and the quotient will be the present worth of the first year's annuity.

---

* The reason of this rule is evident from the nature of the question, and what was said on the same subject in the purchasing of annuities at Simple Interest.

Let p = present worth of the annuity, and the other letters as before, then as the amount $=\frac{r^t-1}{r-1}\times n$, and as the present worth or principal of this, according to the principles of Compound Interest, is the amount divided by r^t, therefore

$$n\times\frac{r^t-1}{r^{t+1}-r^t}=p, \text{ and } p\times\frac{r^{t+1}-r^t}{r^t-1}=n.$$

And from these theorems all the cases, where the purchase

2. Divide the annuity by the square of the ratio, and the quotient will be the present worth of the annuity for the second year.

---

of annuities is concerned, may be exhibited in logarithmic terms, as follows.

I. $Log.\ n + Log.\ \overline{1-\frac{1}{r}} - Log.\ \overline{r-1} = Log.\ p.$

II. $Log.\ p + Log.\ \overline{r-1} - Log.\ \overline{1-\frac{1}{r^t}} = Log.\ n.$

III. $\frac{Log.\ n - Log.\ \overline{n+p-pr}}{Log.\ r} = t.$ IV. $r^{t}+1-\overline{\frac{n}{p}+1}\times rt+\frac{n}{p}=0.$

Let t express the number of half years or quarters, n the half year's or quarter's payment, and r the sum of one pound and $\frac{1}{2}$ or $\frac{1}{4}$ year's interest, then all the preceding rules are applicable to half-yearly and quarterly payments, the same as to whole years.

The amount of an annuity may also be found for years and parts of a year thus:

1. Find the amount for the whole years as before.

2. Find the interest of that amount for the given parts of a year.

3. Add this interest to the former amount, and it will give the whole amount required.

The present worth of an annuity for years and parts of a year may be found thus:

1. Find the present worth for the whole years as before.

2. Find the present worth of this present worth, discounting for the given parts of a year, and it will be the whole present worth required.

3. Find, in like manner, the present worth of each year by itself, and the sum of all these will be the value of the annuity sought.

EXAMPLES.

1. What is the present worth of an annuity of 40l. to continue 5 years, discounting at 5 per cent. per annum, compound interest?

ratio =	1·05)40·00000(38·095=	present worth for first [year.
$\overline{\text{ratio}}\|^2$ =	1·1025)40·00000(36·281=	do. for 2d year.
$\overline{\text{ratio}}\|^3$ =	1·157525)40·00000(34·556=	do. for 3d year.
$\overline{\text{ratio}}\|^4$ =	1·215506)40·00000(32·899=	do. for 4th year.
$\overline{\text{ratio}}\|^5$ =	1·276218)40·00000(31·342=	do. for 5th year.

173·173 = 173l. 3s. 5½d. = whole present worth of the annuity required.

2. What is the present worth of an annuity of 21l. 10s. 9¼d. to continue 7 years, at 6 per cent. per annum, compound interest? Ans. 120l. 5s.

3. What is 70l. per annum, to continue 59 years, worth in present money, at the rate of 5 per cent. per annum?

Ans. 1321·3021l.

To find the present worth of a Freehold Estate, or an Annuity to continue forever, at Compound Interest.

RULE.*

As the rate per cent. is to 100l. so is the yearly rent to the value required.

---

* The reason of this rule is obvious; for since a year's interest of the price, which is given for it, is the annuity, there can

EXAMPLES.

1. An estate brings in yearly 79l. 4s. what would it sell for, allowing the purchaser $4\frac{1}{2}$ per cent, compound interest for his money?

$$4.5 : 100 :: 79.2 :$$

```
            100
        ----------
   4.5)7920.0(1760l. the answer.
       45
       ---
       342
       315
       ---
        270
        270
        ---
```

---

neither more nor less be made of that price than of the annuity, whether it be employed at simple or compound interest.

The same thing may be shown thus: the present worth of an annuity to continue forever is $\frac{n}{r}+\frac{n}{r^2}+\frac{n}{r^3}+\frac{n}{r^4}$, &c. *ad infinitum*, as has been shown before; but the sum of this series, by the rules of Geometrical Progression, is $\frac{n}{r-1}$; therefore $r-1 : 1 :: n : \frac{n}{r-1}$, which is the rule.

The following theorems show all the varieties of this rule.

I. $\frac{n}{r-1}=p$. II. $\overline{r-1}\times p=n$. III. $\frac{n}{p}+1=r$, or $\frac{n}{p}=r-1$.

The price of a freehold estate, or an annuity to continue forever, at simple interest, would be expressed by $\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}+\frac{1}{1+4r}$, &c. *ad infinitum*; but the sum of this series is infinite, or greater than any assignable number, which sufficiently shows the absurdity of using simple interest in these cases.

2. What is the price of a perpetual annuity of 40l. discounting at 5 per cent. compound interest? Ans. 800l.

3. What is a freehold estate of 75l. a year worth, allowing the buyer 6 per cent. compound interest for his money? Ans. 1250l.

To find the present worth of an Annuity, or Freehold Estate, in Reversion, at Compound Interest.

RULE.*

1. Find the present worth of the annuity, as if it were to be entered on immediately.

2. Find the present worth of the last present worth, discounting for the time between the purchase and commencement of the annuity, and it will be the answer required.

EXAMPLES.

1. The reversion of a freehold estate of 79l. 4s. per annum to commence 7 years hence, is to be sold: what is it worth in ready money, allowing the purchaser 4½ per cent. for his money?

---

* This rule is sufficiently evident without a demonstration.

Those, who wish to be acquainted with the manner of computing the values of annuities on lives, may consult the writings of Mr. Demoivre, Mr. Simpson, and Dr. Price, all of whom have handled this subject in a very skilful and masterly manner.

Dr. Price's Treatise on Annuities and Revolutionary Payments is an excellent performance, and will be found a very valuable acquisition to those, whose inclinations lead them to studies of this nature.

4·5 : 100 : : 79·2 :
100

4·5)7920·0(1760 = present worth, if entered on immediately.
45

342
315

270
270

0

and $\overline{1{\cdot}045|}^{7}$ = 1·360862)1760·000(1293·297 = 1293l. 5s. 11¼d. = present worth of 1760l. for 7 years, or the whole present worth required.

2. Which is most advantageous, a term of 15 years in an estate of 100l. per annum, or the reversion of such an estate forever, after the expiration of the said 15 years, computing at the rate of 5 per cent. per annum, compound interest?

Ans. The first term of 15 years is better than the reversion forever afterward, by 75l. 18s. 7½d.

3. Suppose I would add 5 years to a running lease of 15 years to come, the improved rent being 186l. 7s. 6d. per annum; what ought I to lay down for this favour, discounting at 4 per cent. per annum, compound interest?

Ans. 460l. 14s. 1¾d.

POSITION.

POSITION is a method of performing such questions, as cannot be resolved by the common direct rules, and is of two kinds, called *single* and *double*.

SINGLE POSITION.

Single Position teaches to resolve those questions, whose results are proportional to their suppositions.

RULE.*

1. Take any number and perform the same operations with it, as are described to be performed in the question.

2. Then say, as the result of the operation is to the position, so is the result in the question to the number required.

---

* Such questions properly belong to this rule, as require the multiplication or division of the number sought by any proposed number; or when it is to be increased or diminished by itself, or any parts of itself, a certain proposed number of times. For in this case the reason of the rule is obvious; it being then evident, that the results are proportional to the suppositions.

Thus $\begin{cases} nx : x :: na : a \\ \frac{x}{n} : x :: \frac{a}{n} : a \\ \frac{x}{n} \pm \frac{x}{m}, \text{\&c.} : x :: \frac{a}{n} \pm \frac{a}{m}, \text{\&c.} : a, \text{ and so on.} \end{cases}$

NOTE. 1 may be made a constant supposition in all questions; and in most cases it is better than any other number.

EXAMPLES.

1. A's age is double that of B, and B's is triple that of C, and the sum of all their ages is 140: what is each person's age?

Suppose A's age to be 60
Then will B's = $\frac{60}{2}$ = 30
And C's = $\frac{30}{3}$ = 10

100 sum.

As 100 : 60 :: 140 : $\frac{140\times60}{100}$ = 84=A's age.
Consequently $\frac{84}{2}$ = 42=B's.
And $\frac{42}{3}$ = 14=C's.

140 Proof.

2. A certain sum of money is to be divided between 4 persons, in such a manner, that the first shall have $\frac{1}{3}$ of it; the second $\frac{1}{4}$; the third $\frac{1}{6}$; and the fourth the remainder, which is 28l.: what is the sum? Ans. 112l.

3. A person, after spending $\frac{1}{3}$ and $\frac{1}{4}$ of his money, had 60l. left: what had he at first? Ans. 144l.

4. What number is that, which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself, the sum shall be 125? Ans. 60.

5. A person bought a chaise, horse, and harness for 60l.; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness: what did he give for each?

Ans. 13l. 6s. 8d. for the horse, 6l. 13s. 4d. for the harness, and 40l. for the chaise.

6. A vessel has 3 cocks, A, B, and C; A can fill it in 1 hour, B in 2, and C in 3: in what time will they all fill it together? Ans. $\frac{6}{11}$ hour.

DOUBLE POSITION.

Double Position teaches to resolve questions by making two suppositions of false numbers.

RULE.*

1. Take any two convenient numbers, and proceed with each according to the conditions of the question.

---

* The rule is founded on this supposition, that the first error is to the second, as the difference between the true and first supposed number is to the difference between the true and second supposed number: when that is not the case, the exact answer to the question cannot be found by this rule.

That the rule is true, according to the supposition, may be thus demonstrated.

Let A and B be any two numbers, produced from a and b by similar operations: it is required to find the number, from which N is produced by a like operation.

Put $x =$ number required, and let $N - A = r$, and $N - B = s$.

Then according to the supposition, on which the rule is founded, $r : s :: x - a : x - b$, whence, by multiplying means and extremes, $rx - rb = sx - sa$; and, by transposition, $rx - sx = rb - sa$; and, by division, $x = \frac{rb - sa}{r - s} =$ number sought.

Again, if r and s be both negative, we shall have $-r : -s :: x - a : x - b$, and therefore $-rx + rb = -sx + sa$; and $rx - sx = rb - sa$; whence $x = \frac{rb - sa}{r - s}$, as before.

In like manner, if r or s be negative, we shall have $x = \frac{rb + sa}{r + s}$, by working as before, which is the rule.

NOTE. It will be often advantageous to make 1 and 0 the suppositions.

2. Find how much the results are different from the result in the question.

3. Multiply each of the errors by the contrary supposition, and find the sum or difference of the products.

4. If the errors be alike, divide the difference of the products by the difference of the errors, and the quotient will be the answer.

5. If the errors be unlike, divide the sum of the products by the sum of the errors, and the quotient will be the answer.

NOTE. The errors are said to be *alike*, when they are both too great or both too little; and *unlike*, when one is too great and the other too little.

EXAMPLES.

1. A lady bought tabby at 4s. a yard, and Persian at 2s. a yard; the whole number of yards she bought was 8, and the whole price 20s.: how many yards had she of each sort?

Suppose 4 yards of tabby, value	16s.
Then she must have 4 yards of Persian, value	8
Sum of their values	24
So that the first error is +	4
Again, suppose she had 3 yards of tabby at	12s.
Then she must have 5 yards of Persian at	10
Sum of their values	22
So that the second error is +	2

Then $4-2=2=$ difference of the errors.

Also $4\times2=8=$ product of the first supposition and second error.

And $3\times4=12=$ product of the second supposition by the first error.

And $12-8=4=$ their difference.

Whence $4\div2=2=$ yards of tabby, } the answer.
And $8-2=6=$ yards of Persian, }

2. Two persons, A and B, have both the same income; A saves $\frac{1}{5}$ of his yearly; but B, by spending 50l. per annum more than A, at the end of 4 years finds himself 100l. in debt: what is their income, and what do they spend per annum?

Ans. Their income is 125l. per annum; A spends 100l. and B 150l. per annum.

3. Two persons, A and B, lay out equal sums of money in trade; A gains 126l. and B loses 87l. and A's money is now double that of B: what did each lay out? Ans. 300l.

4. A laborer was hired for 40 days, on this condition, that he should receive 20d. for every day he wrought, and forfeit 10d. for every day he was idle; now he received at last 2l. 1s. 8d.: how many days did he work, and how many was he idle?

Ans. He wrought 30 days, and was idle 10.

5. A gentleman has two horses of considerable value, and a saddle worth 50l.; now, if the saddle be put on the back of the first horse, it will make his value double that of the second; but if it be put on the back of the second, it will make his value triple that of the first: what is the value of each horse? Ans. One 30l. and the other 40l.

6. There is a fish, whose head is 9 inches long, and his tail is as long as his head and half as long as his body, and his body is as long as his tail and his head: what is the whole length of the fish? Ans. 6 feet.

PERMUTATION AND COMBINATION.

The *Permutation of Quantities* is the showing how many different ways the order or position of any given number of things may be changed.

This is also called *Variation*, *Alternation*, or *Changes*; and the only thing to be regarded here is the order they stand in; for no two parcels are to have all their quantities placed in the same situation.

The *Combination of Quantities* is the showing how often a less number of things can be taken out of a greater, and combined together, without considering their places, or the order they stand in.

This is sometimes called *Election*, or *Choice*; and here every parcel must be different from all the rest, and no two are to have precisely the same quantities, or things.

The *Composition of Quantities* is the taking a given number of quantities out of as many equal rows of different quantities, one out of each row, and combining them together.

Here no regard is had to their places; and it differs from combination only, as that admits of but one row, or set of things.

Combination of the same form are those, in which there is the same number of quantities, and the same repetitions: thus, *abcc*, *bbad*, *deef*, &c. are of the same form; but *abbc*, *abbb*, *aacc*, &c. are of different forms.

PROBLEM 1.

To find the number of permutations, or changes, that can be made of any given number of things, all different from each other.

RULE.*

Multiply all the terms of the natural series of numbers, from 1 up to the given number, continually together, and the last product will be the answer required.

EXAMPLES.

1. How many changes may be made with these three letters, *abc*?

$$\begin{array}{r} 1 \\ 2 \\ \hline 2 \\ 3 \\ \hline 6 \end{array}$$

Or $1 \times 2 \times 3 = 6$ the answer.

---

* The reason of the rule may be shown thus: any one thing is capable only of one position, as *a*.

Any two things, *a* and *b*, are only capable of two variations; as *ab*, *ba*; whose number is expressed by 1×2.

If there be 3 things, *a*, *b*, and *c*, then any two of them, leaving out the third, will have 1×2 variations; and consequently, when the third is taken in, there will be $1 \times 2 \times 3$ variations.

In the same manner, when there are 4 things, every 3, leaving out the forth, will have $1 \times 2 \times 3$ variations. Then, the fourth being taken in, there will be $1 \times 2 \times 3 \times 4$ variations. And so on, as far as you please.

the changes.

abc
acb
bac
bca
cab
cba

2. How many changes may be rung on 6 bells?

Ans. 720.

3. For how many days can 7 persons be placed in a different position at dinner? Ans. 5040 days.

4. How many changes may be rung on 12 bells, and how long would they be in ringing, supposing 10 changes to be rung in 1 minute, and the year to consist of 365 days 5 hours and 49 minutes?

Ans. 479001600 changes, and 91y. 26d. 22h. 41m.

5. How many changes may be made of the words in the following verse? *Tot tibi sunt dotes, virgo, quot sydera cælo.*

Ans. 40320.

PROBLEM 2.

Any number of different things being given, to find how many changes can be made out of them, by taking any given number at a time.

RULE.*

Take a series of numbers, beginning at the number of things given, and decreasing by 1 till the number of terms

* This rule, expressed in terms, is as follows: $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}$, &c. to n terms; where m = number of things given, and n = quantities to be taken at a time.

be equal to the number of things to be taken at a time, and the product of all the terms will be the answer required.

In order to demonstrate the rule, it will be necessary to premise the following

LEMMA.

The number of changes of m things, taken n at a time, is equal to m changes of $m-1$ things, taken $n-1$ at a time.

DEMONSTRATION. Let any 5 quantities, *abcde*, be given.

First, leave out the a, and let v = number of all the variations of every two, *bc*, *bd*, &c. that can be taken out of the 4 remaining quantities, *bcde*.

Now let a be put in the first place of each of them, *abc*, *abd*, &c. and the number of changes will still remain the same; that is, v = number of variatons of every 3 out of the 5, *abcde*, when a is first.

In like manner, if b, c, d, e, be successively left out, the number of variations of all the twos will also $= v$; and b, c, d, e, being respectively put in the first place, to make 3 quantities out of 5, there will still be v variations as before.

But these are all the variations, that can happen of 3 things out of 5, when a, b, c, d, e, are successively put first; and therefore the sum of all these is the sum of all the changes of 3 things out of 5.

But the sum of these is so many times v, as is the number of things; that is, $5v$, or mv, = all the changes of three things out of 5. And the same way of reasoning may be applied to any numbers whatever.

DEMONSTRATION OF THE RULE. Let any 7 things, *abcdefg*, be given, and let 3 be the number of quantities to be taken.

Then $m=7$, and $n=3$.

EXAMPLES.

1. How many changes may be made out of the 3 letters, *abc*, by taking 2 at a time.

3
2
—
6

Or $3\times2=6$ the answer.

The changes.

ab
ba
ac
ca
bc
cb

2. How many words can be made with 5 letters of the alphabet, it being admitted, that a number of consonants may make a word? Ans. 5100480.

PROBLEM 3.

Any number of things being given, whereof there are several given things of one sort, several of another, &c. to find how many changes can be made out of them all.

Now it is evident, that the number of changes, that can be made by taking 1 by 1 out of 5 things, will be 5, which let $=v$.

Then, by the lemma, when $m=6$ and $n=2$, the number of changes will $=mv=6\times5$; which let $=v$ a second time.

Again by lemma, when $m=7$ and $n=3$, the number of changes $=mv=7\times6\times5$; that is, $m\,v=m\times\overline{m-1}\times\overline{m-2}$, continued to 3, or n terms. And the same may be shown for any other numbers.

RULE.*

1. Take the series 1, 2, 3, 4, &c. up to the number of things given, and find the product of all the terms.

---

* This rule is expressed in terms thus: $\frac{1\times2\times3\times4\times5, \text{ \&c. to } m}{1\times2\times3, \text{ \&c. to } p \times 1\times2\times3, \text{ \&c. to } q, \text{ \&c.}}$; where m = number of things given, p = number of things of the first sort, q = number of things of the second sort, &c.

The DEMONSTRATION may be shown as follows.

Any two quantities, a, b, both different, admit of 2 changes; but if the quantities be the same, or ab become aa, there will be but one alteration, which may be expressed by $\frac{1\times2}{1\times2}=1$.

Any three quantities, abc, all different from each other, afford 6 variations; but if the quantities be all alike, or abc become aaa, then the 6 variations will be reduced to 1, which may be expressed by $\frac{1\times2\times3}{1\times2\times3}=1$. Again, if two of the quantities only be alike, or abc become aac, then the six variations will be reduced to these 3, aac, caa, and aca, which may be expressed by $\frac{1\times2\times3}{1\times2}=3$.

Any four quantities, $abcd$, all different from each other, will admit of 24 variations; but if the quantities be the same, or $abcd$ become $aaaa$, the number of variations will be reduced to one; which is $=\frac{1\times2\times3\times4}{1\times2\times3\times4}=1$. Again, if three of the quantities only be the same, or $abcd$ become $aaab$, the number of variations will be reduced to these 4, $aaab$, $aaba$, $abaa$, and $baaa$, which is $=\frac{1\times2\times3\times4}{1\times2\times3}=4$. And thus it may be shown, that,

2. Take the series 1, 2, 3, 4, &c. up to the number of given things of the first sort, and the series 1, 2, 3, 4, &c. up to the number of given things of the second sort, &c.

3. Divide the product of all the terms of the first series by the joint product of all the terms of the remaining ones, and the quotient will be the answer required.

EXAMPLES.

1. How many variations may be made of the letters in the word *Bacchanalia*?

1×2(=number of *c*s)= 2

$1\times2\times3\times4$(=number of *a*s)= 24

$1\times2\times3\times4\times5\times6\times7\times8\times9\times10\times11$(=number of letters in the word)=39916800.

2×24=48)39916800(831600 the answer.

151

76

288

2. How many different numbers can be made of the following figures, 1220005555? Ans. 12600.

3. What is the variety in the succession of the following musical notes, fa, fa, fa, sol, sol, la, mi, fa? Ans. 3360.

---

if two of the quantities be alike, or the 4 quantities be *aabc*, the number of variations will be reduced to 12, which may be expressed by $\frac{1\times2\times3\times4}{1\times2} = 12$.

And by reasoning in the same manner it will appear, that the number of changes, which can be made of the quantities, *abbccc*, is equal to 60, which may be expressed by $\frac{1\times2\times3\times4\times5\times6}{1\times2\times1\times2\times3} =$ 60; and so of any other quantities whatever.

PROBLEM 4.

To find the changes of any given number of things, taken a given number at a time; in which there are several given things of one sort, several of another, &c.

RULE.*

1. Find all the different forms of combination of all the given things, taken as many at a time as in the question.

2. Find the number of changes in any form, and multiply it by the number of combinations in that form.

3. Do the same for every distinct form; and the sum of all the products will give the whole number of changes required.

NOTE. *To find the different forms of combination proceed thus:*

1. Place the things so, that the greatest indices may be first, and the rest in order.

2. Begin with the first letter, and join it to the second, third, fourth, &c. to the last.

3. Then take the second letter, and join it to the third, fourth, &c. to the last; and so on through the whole, always remembering to reject such combinations as have occurred before; and this will give the combinations of all the twos.

4. Join the first letter to every one of the twos following it; and the second, third, &c. as before; and it will give the combinations of all the threes.

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* The reason of this rule is plain from what has been shown before, and the nature of the problem.

5. Proceed in the same manner to get the combinations of all the fours, &c. and you will at last get all the several forms of combination, and the number in each form.

EXAMPLES.

1. How many changes may be made of every 4 letters, that can be taken out of these 6, *aaabbc*?

No. of forms.	Forms.	No. of combinations.	No. of changes in each form.
1st	a^3b, a^3c	2	$\dfrac{1\times2\times3\times4=24}{1\times2\times3\quad=6}=4.$
2d	a^2b^2	1	$\dfrac{1\times2\times3\times4=24}{1\times2\times1\times2=4}=6.$
3d	a^2bc, b^2ac	2	$\dfrac{1\times2\times3\times4=24}{1\times2\quad=2}=12.$

$4\times2=8$
$6\times1=6$
$12\times2=24$

38 = the number of changes required.

2. How many chunges ean be made of every 8 letters out of these 10, *aaaabbccde*? Ans. 22260.

3. How many different numbers can be made out of 1 unit, 2 twos, 3 threes, 4 fours, and 5 fives, taken 5 at a time? Ans. 2111.

PROBLEM 5.

To find the number of combinations of any given number of things, all different from one another, taken any given number at a time.

RULE.*

1. Take the series 1, 2, 3, 4, &c. up to the number to be taken at a time, and find the prodnct of all the terms.

---

* This rule, expressed algebraically, is $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c. to n terms; where m is the number of given quantities, and n those to be taken at a time.

DEMONSTRATION OF THE RULE. 1. Let the number of things to be taken at a time be 2, and the things to be combined $= m$.

Now, when m, or the number of things to be combined, is only two, as a and b, it is evident, that there can be only one combination, as ab; but if m be increased by 1, or the letters to be combined be 3, as abc, then it is plain, that the number of combinations will be increased by 2, since with each of the former letters, a and b, the new letter c may be joined. It is evident therefore, that the whole number of combinations, in this case, will be truly expressed by $1+2$.

Again, if m be increased by one letter more, or the whole number of letters be four, as $abcd$; then it will appear, that the whole number of combinations must be increased by 3, since with each of the preceding letters the new letter d may be combined. The combinations therefore, in this case, will be truly expressed by $1+2+3$.

In the same manner it may be shown, that the whole number

2. Take a series of as many terms, decreasing by 1 from the given number, out of which the election is to be made, and find the product of all the terms.

---

of combinations of 2, in 5 things, will be $1+2+3+4$; of 2 in 6 things, $1+2+3+4+5$; and of 2, in 7, $1+2+3+4+5+6$, &c.

Whence universally, the number of combinations of m things, taken 2 by 2, is $=1+2+3+4+5+6$, &c. to $\overline{m-1}$ terms.

But the sum of this series is $=\frac{m}{1}+\frac{m-1}{2}$, which is the same as the rule.

2. Let now the number of quantities in each combination be supposed to be three.

Then it is plain, that when $m=3$, or the things to be combined are abc, there can be only one combination; but if m be increased by 1, or the things to be combined be 4, as $abcd$, then will the number of combinations be increased by 3; since 3 is the number of combinations of 2 in all the preceding letters abc, and with each two of these the new letter d may be combined.

The number of combinations therefore, in this case, is $1+3$.

Again, if m be increased by one more, or the number of letters be supposed 5; then the former number of combinations will be increased by 6; that is, by all the combinations of 2 in the 4 preceding letters, $abcd$; since, as before, with each two of these the new letter e may be combined.

The number of combinations therefore, in this case, is $1+3+6$.

Whence universally, the number of combinations of m things, taken 3 by 3, is $1+3+6+10$, &c. to $m-2$ terms.

3. Divide the last product by the former, and the quotient will be the number sought.

EXAMPLES.

1. How many combinations can be made of 6 letters out of ten?

1×2×3×4×5×6(=the number to be taken at a time)=720
10×9×8×7×6×5(=same number from 10)=151200

```
720)151200(210 the answer.
    1440
    ----
      720
      720
      ---
        0
```

2. How many combinations can be made of 2 letters out of 24 letters of the alphabet? Ans. 276.

3. A general, who had often been succesful in war, was asked by his King, what reward he should confer on him for his services; the general only desired a farthing for every file of 10 men in a file, which he could make with a body of 100 men: what is the amount in pounds sterling?

Ans. 1803157235Ol. 9s. 2d.

But the sum of this series is $=\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$, which is the same as the rule.

And the same thing will hold, let the number of things, to be taken at a time, be what it may; therefore the number of combinations of m things, taken n at a time, will $=\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$ $\times\frac{m-3}{4}$, &c. to n terms. Q. E. D.

PROBLEM 6.

To find the number of combinations of any given number of things, by taking any given number at a time; in which there are several things of one sort, several of another, &c.

RULE.

1. Find by trial the number of different forms, which the things, to be taken at a time, will admit of, and the number of combinations in each.

2. Add together all the combinations, thus found, and the sum will be the number required.

EXAMPLES.

1. Let the things proposed be *aaabbc*; it is required to find the number of combinations, that can be made of every three of these quantities.

Forms.	Combinations.
a^3	1
a^2b, a^2c, b^2a, b^2c	4
abc	1
	6=number of combinations required.

2. Let *aaabbbcc* be proposed; it is required to find the number of combinations of these quantities, taken 4 at a time? Ans. 10.

3. How many combinations are there in *aaaabbccde*, 8 being taken at a time? Ans. 13.

4. How many combinations are there in *aaaaabbbbbccccdd ddeeeeffffg*, 10 being taken at a time? Ans. 2819.

PROBLEM 7.

To find the compositions of any number, in an equal number of sets, the things themselves being all different.

RULE.*

Multiply the number of things in every set continually together, and the product will be the answer required.

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* DEMONSTRATION. Suppose there are only two sets; then it is plain, that every quantity of one set, being combined with every quantity of the other, will make all the compositions of two things, in these two sets; and the number of these compositions is evidently the product of the number of quantities in one set by that in the other.

Again, suppose there are three sets; then the composition of two, in any two of the sets, being combined with every quantity of the third, will make all the compositions of 3 in the 3 sets. That is, the compositions of 2 in any two of the sets, being multiplied by the number of quantities in the remaining set, will produce the compositions of 3 in the 3 sets; which is evidently the continual product of all the 3 numbers in the 3 sets. And the same manner of reasoning will hold, let the number of sets be what it will. Q. E. D.

The doctrine of permutations, combinations, &c. is of very extensive use in different parts of the mathematics; particularly in the calculations of annuities and chances. The subject might have been pursued to a much greater length; but what has been done already will be found sufficient for most of the purposes, to which things of this nature are applicable.

EXAMPLES.

1. Suppose there are 4 companies, in each of which there are 9 men; it is required to find how many ways 4 men may be chosen, one out of each company.

```
   9
   9
  --
  81
   9
 ---
 729
   9
----
6561
```

Or, 9×9×9×9=6561 the answer.

2. Suppose there are 4 companies, in one of which there are 6 men, in another 8, and in each of the other two 9; what are the choices, by a composition of 4 men, one out of each company? Ans. 3888.

3. How many changes are there in throwing 5 dice? Ans. 7776.

MISCELLANEOUS QUESTIONS.

1. WHAT difference is there between twice five and twenty, and twice twenty-five? Ans. 20.

2. A was born when B was 21 years of age; how old will A be when B is 47; and what will be the age of B when A is 60? Ans. A 26, B 81.

3. What number, taken from the square of 48, will leave 16 times 54? Ans. 1440.

4. What number, added to the thirty-first part of 3813, will make the sum 200? Ans. 77.

5. The remainder of a division is 325, the quotient 467, and the divisor is 43 more than the sum of both: what is the dividend? Ans. 390270.

6. Two persons depart from the same place at the same time; the one travels 30, the other 35 miles a day: how far are they distant at the end of 7 days, if they travel both the same road; and how far, if they travel in contrary directions? Ans. 35, and 455 miles.

7. A tradesman increased his estate annually by 100l. more than $\frac{1}{4}$ part of it, and at the end of 4 years found, that his estate amounted to 10342l. 3s. 9d. What had he at first? Ans. 4000l.

8. Divide 1200 acres of land among A, B, and C, so that B may have 100 more than A, and C 64 more than B.
Ans. A 312, B 412, and C 476.

9. Divide 1000 crowns; give A 120 more, and B 95 less, than C. Ans. A 445, B 230, C 325.

10. What sum of money will amount to 132l. 16s. 3d. in 15 months, at 5 per cent. per annum, simple interest?
Ans. 125l.

11. A father divided his fortune among his sons, giving A 4 as often as B 3, and C 5 as often as B 6; what was the whole legacy, supposing A's share 5000l.? Ans. 11875l.

12. If 1000 men, besieged in a town with provisions for 5 weeks, each man being allowed 16oz. a day, were rein-

forced with 500 men more. On hearing, that they cannot be relieved till the end of 8 weeks, how many ounces a day must each man have, that the provision may last that time?

Ans. $6\frac{2}{3}$ oz.

13. What number is that, to which if $\frac{2}{7}$ of $\frac{5}{9}$ be added, the sum will be 1? Ans. $\frac{53}{63}$.

14. A father dying left his son a fortune, $\frac{1}{4}$ of which he ran through in 8 months; $\frac{3}{7}$ of the remainder lasted him twelve months longer; after which he had only 410l. left. What did his father bequeath him? Ans. 956l. 13s. 4d.

15. A guardian paid his ward 3500l. for 2500l. which he had in his hands 8 years. What rate of interest did he allow him? Ans. 5 per cent.

16. A person, being asked the hour of the day, said, the time past noon is equal to $\frac{4}{5}$ of the time till midnight. What was the time? Ans. 20 min. past 5.

17. A person, looking on his watch, was asked, what was the time of the day; he answered, it is between 4 and 5; but a more particular answer being required, he said, that the hour and minute hands were then exactly together. What was the time? Ans. $21\frac{9}{11}$ minutes past 4.

18. With 12 gallons of Canary, at 6s. 4d. a gallon, I mixed 18 gallons of white wine, at 4s. 10d. a gal. and 12 gallons of cider, at 3s. 1d. a gal. At what rate must I sell a quart of this composition, so as to clear 10 per cent?

Ans. 1s. $3\frac{5}{7}$d.

19. What length must be cut off a board, $8\frac{3}{8}$ inches broad, to contain a square foot, or as much as 12 inches in length and 12 in breadth? Ans. $17\frac{13}{67}$ in.

20. What difference is there between the interest of 350l. at 4 per cent. for 8 years, and the discount of the same sum at the same rate and for the same time? Ans. 27l. $3\frac{1}{33}$s.

21. A father devised $\frac{7}{18}$ of his estate to one of his sons, and $\frac{7}{18}$ of the residue to another, and the surplus to his relict for life; the children's legacies were found to be 257l. 3s. 4d. different. What money did he leave for the widow?

Ans. 635l. $10\frac{30}{49}$d.

22. What number is that, from which if you take $\frac{2}{7}$ of $\frac{3}{8}$, and to the remainder add $\frac{7}{16}$ of $\frac{1}{20}$, the sum will be 10?

Ans. $10\frac{191}{2240}$.

23. A man dying left his wife in expectation, that a child would be afterward added to the surviving family; and making his will ordered, that, if the child were a son, $\frac{2}{3}$ of his estate should belong to him, and the remainder to his mother; but if it were a daughter, he appointed the mother $\frac{2}{3}$, and the child the remainder. But it happened, that the addition was both a son and a daughter, by which the widow lost in equity 2400l. more than if there had been only a girl. What would have been her dowry, had she had only a son?

Ans. 2100l.

24. A young hare starts 40 yards before a grey-hound, and is not perceived by him till she has been up 40 seconds; she scuds away at the rate of ten miles an hour, and the dog, on view, makes after her at the rate of 18. How long will

the course continue, and what will be the length of it from the place, where the dog set out?

Ans. $60\frac{5}{23}$ seconds, and 530 yards run.

25. A reservoir for water has two cocks to supply it; by the first alone it may be filled in 40 minutes, by the second in 50 minutes, and it has a discharging cock, by which it may, when full, be emptied in 25 minutes. Now supposing, that these three cocks are all left open, that the water comes in, and that the influx and efflux of the water are always alike, in what time would the cistern be filled?

Ans. 3 hours 20 min.

26. A sets out from London for Lincoln precisely at the time, when B at Lincoln sets forward for London, distant 100 miles; after 7 hours they met on the road, and it then appeared, that A had ridden $1\frac{1}{2}$ mile an hour more than B. At what rate an hour did each of them travel?

Ans. A $7\frac{25}{28}$, B $6\frac{11}{28}$ miles.

27. What part of 3d. is a third part of 2d. Ans. $\frac{2}{9}$.

28. A has by him $1\frac{1}{2}$cwt. of tea, the prime cost of which was 96l. sterling. Now granting interest to be at 5 per cent. it is required to find how he must rate it per pound to B, so that by taking his negotiable note, payable at 3 months, he may clear 20 guineas by the bargain?

Ans. 14s. $1\frac{13}{59}$d. sterling.

29. What annuity is sufficient to pay off 50 millions of pounds in 30 years, at 4 per cent. compound interest?

Ans. 2891505l.

30. There is an island 73 miles in circumference, and 3 footmen all start together to travel the same way about it; A goes 5 miles a day, B 8, and C 10; when will they all come together again? Ans. 73 days.

31. A man, being asked how many sheep he had in his drove, said, if he had as many more, half as many more, and 7 sheep and a half, he should have 20: how many had he? Ans. 5.

32. A person left 40s. to 4 poor widows, A, B, C, and D; to A he left $\frac{1}{3}$, to B $\frac{1}{4}$, to C $\frac{1}{5}$, and to D $\frac{1}{6}$, desiring the whole might be distributed accordingly; what is the proper share of each?

Ans. A's share 14s. $\frac{16}{38}$d. B's 10s. $6\frac{12}{38}$d. C's 8s. $5\frac{10}{38}$d. D's 7s. $\frac{8}{38}$d.

33. A general, disposing of his army into a square, finds he has 284 soldiers over and above; but increasing each side with one soldier, he wants 25 to fill up the square; how many soldiers had he? Ans. 24000.

34. There is a prize of 212l. 14s. 7d. to be divided among a captain, 4 men, and a boy; the captain is to have a share and a half; the men each a share, and the boy $\frac{1}{3}$ of a share: what ought each person to have?

Ans. The captain 54l. 14s. $\frac{3}{7}$d. each man 36l. 9s. $4\frac{2}{7}$d. and the boy 12l. 3s. $1\frac{3}{7}$d.

35. A cistern, containing 60 gallons of water, has 3 unequal cocks for discharging it; the greatest cock will empty it in one hour, the second in 2 hours, and the third in 3: in what time will it be empty, if they all run together?

Ans. $32\frac{8}{11}$ minutes.

36. In an orchard of fruit trees, $\frac{1}{2}$ of them bear apples, $\frac{1}{4}$ pears, $\frac{1}{6}$ plumbs, and 50 of them cherries: how many trees are there in all? Ans. 600.

37. A can do a piece of work alone in ten days, and B in 13; if both be set about it together, in what time will it be finished? Ans. $5\frac{15}{23}$ days.

38. A, B, and C are to share 100000l. in the proportion of $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, respectively; but C's part being lost by his death, it is required to divide the whole sum properly between the other two.

Ans. A's part is $57142\frac{283}{330}$, and B's $42857\frac{47}{330}$

END.